REFLECTIONS ON THE NATURE OF MATHEMATICS
Mathematics, the physical universe and biology
(INCOMPLETE DRAFT: Liable to change)
(Comments and criticisms welcome)

Aaron Sloman

http://www.cs.bham.ac.uk/~axs/

School of Computer Science, University of Birmingham

(Philosopher in a Computer Science department)

NOTE: some of these ideas are being revised and extended in
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/maths-multiple-foundations.html

This document is
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/what-is-maths.html
It is not yet indexed on this web site as there are likely to be changes in response to criticisms.
A temporary PDF version is
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/what-is-maths.pdf

A partial index of discussion notes is in
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/AREADME.html

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The mathematical creativity of evolutionary mechanisms
(Modified version of previous "Extended abstract".)

Many mechanisms now used in reproduction, growth and development could not have existed on this planet when it first formed because they depend on previous products of evolution and in some cases by-products, such as an oxygen rich atmosphere. Those earlier products also depended on earlier products, except for the very first products whose predecessors must have been produced by physical/chemical processes other than biological reproduction. So evolution has not been a uniform process.

All of this required the universe from its very beginnings, or relatively early stages if there was no beginning, to include a *construction-kit* with the potential to continually produce branching successions of new more sophisticated *derived construction kits* able to produce increasingly complex life forms, and new types of construction kit repeatedly adding to the variety and complexity of life forms. This process must have a branching, multi-layered, mathematical structure whose root has a mathematical structure capable of generating all the others, most of them only indirectly, via intermediate construction kits. This can be loosely compared with a mathematical system including formation rules, axioms, inference rules, proofs, lemmas, and theorems. However, the biological theorems concern what is possible in the universe, not what must be true.

> I tried to defend the importance of explanations of possibilities in science in Chapter 2 of Sloman(1978). This required modifications of the theories about science presented by Popper (emphasising falsifiability) and Lakatos (emphasising a distinction between progressive and degenerating scientific research programs).

Implications regarding foundations of mathematics

This has implications regarding both acceptable answers to "What is mathematics?" and "What makes evolution possible?" In particular, if humans are among the products of such mathematical powers in the physical universe, then mathematics cannot all be a product of human thoughts and activities, even if humans were the first to notice their own use of mathematical structures and then to attend to, describe, reason about, and teach others about the powers and limitations of such structures.

Borrowing ideas from James Gibson [1966] and [1979] we can say that natural selection makes use of increasingly complex affordances for biological evolution, produced by its earlier selections and other non-biological processes, e.g. by-products of tides, climate changes, volcanic processes and asteroid impacts. Gibson drew attention to ways in which animal perception processes acquire information about the context-dependent affordances for action provided by the environment.

Not all affordances that exist are perceived, and not all that are perceived are used -- intelligent organisms can make choices between affordances. Moreover, not all actions based on perceived affordances are successful, e.g. because of some unnoticed threat, or because opportunities are not used well.

Over time, natural selection also depends on increasingly complex and varied "affordances for evolutionary change" between which selections are made (mostly blindly), in some cases producing new positive or negative affordances for the organisms concerned, or for their predators, prey, or symbionts.

All of this, combined with the enormous diversity of life forms now on earth, implies that the universe must have had mathematically very deep constructive/inventive/creative powers long before there were any humans. Humans and their creations are among the *products* of those original mathematical powers of the physical universe in combination with natural selection.

The creativity, of the universe in combination with natural selection, now partly mediated by human activities, exceeds the creativity of anything else in past present or future states of the universe.

This implies that long before any humans discovered mathematical truths (theorems) and found increasingly complex proofs of increasingly complex theorems (e.g. in arithmetic and geometry) based on relatively simple starting assumptions, the universe was doing something similar using natural selection in combination with increasingly varied construction kits, all based on the fundamental construction kit and and in most cases intermediate derived construction kits.

The consequences of these processes continually produce new options for selection, at all levels of complexity of life forms.

Over the last two centuries logicians, mathematicians, language theorists, and computer scientists have increasingly made explicit use of mathematical structures with generative powers, including axioms and inference mechanisms used in logic and mathematics, generative grammars used in logic and linguistics and computer science, and programming tools and mechanisms in computer science and software engineering. Theoretical chemists and chemical engineers have been doing something similar on the basis of many partly disconnected fragments of chemical theory concerning possible forms of molecule and possible molecular processes. Newtonian mechanics can also be seen as a generative system whose widespread manifestations even inspired a poet

> Nature and Nature’s laws lay hid in night:
> God said, "Let Newton be!" and all was light.
> Alexander Pope, Epitaph intended for Sir Isaac Newton, 1726, but not used.

New layers of construction kit continually provide new options between which natural selection can choose. Then some of the products of natural selection that take the form of new species, or new physical features or new behaviours in existing species, contribute to new layers of construction kit.

For example, if members of species A or their products are essential foods for members of species B, then the A-individuals are among the construction kits used by B, though later evolved descendents of B may be less dependent on A, because alternative suppliers of required sources of energy or materials for maintenance or growth have evolved independently.

Symbiotic dependence between species may lead to parallel evolution of new mutually dependent products of old evolved construction kits. It may also leed to evolution of new mutually dependent construction kits.

It follows from all this that the theory of natural selection alone cannot explain how the physical universe produced all life forms. Biological evolution uses a selection mechanism, but selection is impossible without options to select from, which vary over time, and depend on the original construction kit and increasingly complex evolved construction kits producing new options.

In particular, natural selection in itself does not explain what creates new options to select from. So a widely accepted answer to the question "How do new forms of life evolve?" usually attributed to Charles Darwin and Alfred Wallace, tersely summarised by Graham Bell (2008) must be wrong (at least in this form):

> *"Living complexity cannot be explained except through selection and does not require any other category of explanation whatsoever."*

The early universe with none of the present life forms must have had the *potential* to produce the huge variety of extremely varied life forms that have ever existed, and many more that have never existed, but could have. The existence of that potential is not explained by natural selection, though natural selection is crucial to the realisation of much of that potential.

Moreover, it needs not only the potential to produce changes (including new biological mechanisms and new species using those mechanisms) but also the potential to produce mechanisms that can prevent some changes -- mechanisms that implement constraints. For example, evolution's ability to produce forests in which orangutans forage in trees requires trees composed of different materials with different properties, some of which prevent the weight of the orangutan bringing the tree down while other parts can easily be broken, chewed, swallowed digested and used to serve a host of biological functions in the animal. Evolved constraints can be used for purposes unrelated to the selection processes that produce them. More familiar examples of evolved preventive mechanisms include camouflage and immune mechanisms.

Before life existed, the physical universe must have had vast generative powers -- the ability to produce an unthinkably large variety of static and changing physical forms, along with a comparably large collection of constraints on those forms. That needs an initial mixture of possibilities and impossibilities (constraints) that is typical of a mathematical space of structures. A familiar example is 2-D or 3-D Euclidean geometry, which specifies an unbounded space of (arbitrarily complex) possible configurations of structures and processes along with impossibilities: e.g. it is impossible for a circle and a square to intersect in exactly 10 points. (This formulation is meant to exclude the interiors of either: it's just their boundaries that have this limitation.)

> *That is a special case of a more general class of geometrical impossibilities whose formulation is left as an exercise for the reader, who need not be a professional mathematician.*

The theory of evolution by natural selection needs to be supplemented by a specification of a *construction kit* with the power to generate all the possibilities from which selections have been made or could have been made or will be made in the future, and all the constraints on those possibilities (i.e. impossibilities) required for the functioning of successful instances.

A specification of a set of possibilities with limitations or constraints (sometimes called "formation rules", or "syntax"), is a specification of a mathematical domain. Discovering what is and is not possible in a precisely specified domain with generative powers can be a very difficult mathematical challenge. The generative domain of positive integers (sometimes called "natural numbers") is very well understood -- but is rich in unanswered questions. E.g. is there a largest integer P such that P is prime, and P+2 is also prime, like the pair 5 and 7, the pair 11 and 13?

If not there must be infinitely many such "twin primes". As far as I know nobody knows yet whether there are or not, or whether the answer is decidable using known means of mathematical reasoning.

The standard theory of natural selection referred to by Bell is inherently incomplete because it lacks a generative specification of the properties of the *Fundamental Construction Kit* (FCK) provided by the physical universe that allowed all the forms of life that have so far existed and all that could exist. That will be a mathematical specification.

So a challenge for physicists working on any proposed "fundamental" physical theory is to show that their theory specifies a construction kit with the right mathematical generative powers and limits. I suspect no current physical theory satisfies that condition, though checking that suspicion will require the conditions to be fleshed out in much more detail than they have been so far: a task that requires multi-disciplinary collaboration between (at least) physics, chemistry, biology, ethology, psychology, neuroscience, linguistics, philosophy, AI, and mathematics.

Physicists can't be expected to know everything that physics needs to explain. They tend to focus on phenomena that can be described using mathematical ideas with which they are familiar, whereas many more types of phenomena are possible or impossible in biology. Of course, from time to time requirements for new physical theories can drive advances in mathematics (e.g. differential and integral calculus, tensor calculus). I am suggesting that more such mathematical advances may be required to serve currently unrecognized needs of fundamental physics, including perhaps a new mathematical theory to describe features of the fundamental construction kit.

An adequate explanatory specification and the demonstrations of its adequacy will be a great mathematical achievement. It may or may not be beyond the mathematical powers of human brains. Moreover, even if we find such a construction-kit specification, and find evidence that it is a correct description of the universe, it could, like the construction kit for the natural numbers, allow the formulation of some fairly simple questions for which answers are unknown, and in some cases may even be undecidable, a complication I'll not pursue in this document, except to note that an example may be whether a physical brain could ever discover a proof or disproof of the twin prime conjecture!

The rest of this document attempts to flesh out those ideas in the context of the Turing-inspired Meta-Morphogenesis (M-M) project, whose main aim is to understand the varieties of types of information processing produced by evolution, and whose root web site is
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/meta-morphogenesis.html

One of the preliminary partial results is the observation that in addition to all the various life forms, evolution needs to produce many *construction kits*, steadily increasing in complexity, used alongside other construction kits that are not products of evolution - e.g. side effects of climate changes, volcanic eruptions, asteroid impacts, or lunar tides.

In many cases, one life form, with needs that cannot be directly addressed by direct products of the fundamental construction kit, uses other life forms as sources or components of their construction kits. That's especially true of humans, who make essential use of bacteria, plants, and other life forms. These are now being extended with designed products of chemical construction kits. (With unknown long term risks?)

A side effect of this line of thinking is to challenge claims that have been made about the nature of mathematics, especially the claim that mathematics is a human creation. The truth is the other way round: humans were created as side effects of mathematically very complex processes generated using the mathematical resources (generative powers) of pre-existing physical mechanisms. Without various mathematical properties of the FCK humans could not have come into existence, although most of the connections are very indirect.

The M-M project includes a very ambitious long term aim, namely to identify the many layers of structure produced by evolution from the FCK, including both physical/chemical/anatomical structures and also information-processing mechanisms produced by evolution and products of evolution. All of those layers will have mathematical features that explain their generative powers.

Human mathematicians may have to extend their theories in order to capture all the biologically relevant properties of the FCK and the processes by which derived construction kits are produced and how they function. It won't be the first time that scientific explanatory requirements have driven expansion of human mathematics.

The achievements of natural selection depended on repeated production of new more sophisticated *derived* construction-kits, all with their own mathematical properties, including construction kits for new information processing subsystems, some of which are virtual machines implemented in older systems.

The use of derived construction kits in which previous products of evolution are stored ready for use is mathematically similar to the use of mathematical algorithms for generating increasingly complex structures that frequently have to use everything previously constructed, as in the Fibonacci sequence, where the Nth number is the sum of previous two Fibonacci numbers, starting with fib(0) = 0 and fib(1) = 1. So computing fib(100) requires computing the two previous values each of which requires its two previous values, and so on. The time required grows surprisingly fast, with very many repeated computations. Programmers have learnt to produce programs that avoid re-computing, and instead simply remember computed values. Then to get the Nth number after computing the previous two numbers all one has to do is add the last two stored values. If they have not yet been computed the mechanism simply descends to the largest stored number and computes the missing intermediate ones, which can produce an enormous saving. These are called "memo-functions", because they use a memory of previous computations.

Likewise evolution does not have to start from scratch to produce a new species because it can use very complex previously evolved construction kits for many of the components, though a new component may require a small modification of a previously evolved construction kit. The readily available derived construction kits (DCKs) are mathematically analogous to stored Memo-function values. This is one of many ways in which evolution made and used important discoveries, that humans had to re-discover in order to continue making progress.

As explained below, the writing of this paper was triggered by an encounter in September 2016 with one of the world’s leading mathematicians, Michael Atiyah, in a conference panel, followed by reading some of what he had written about the nature of mathematics, which I felt did not address some of the features and roles of mathematics that I have been trying to describe. However, there was nothing I read that directly contradicted the points I have been making, except for some suggestions that mathematics is a human creation.

So far, I have given a very high level overview: the rest of this paper is a (messy and incomplete) attempt to fill out some of the missing details, though I think there are still many unsolved problems that will need multi-disciplinary collaboration, including identifying in far more detail what needs to be explained.

The ideas reported here are the latest versions of a succession of increasingly complex attempts to explain the nature of mathematical discovery in a Kantian framework, first presented in my DPhil Thesis (Oxford,1962). The ideas presented here need to be expanded to include a detailed discussion of requirements for designing robots able to develop the same mathematical powers as ancient Greek mathematicians and their precursors. It is not yet clear to me whether digital computers provide a sufficiently rich platform for designing the required virtual machinery. As Turing suggested in 1950, it may turn out that information processing in animal brains make essential use of chemistry. The implications of that possibility are not yet clear.

Background

These notes were written after a panel discussion on the nature of mathematics in Edinburgh on 8th September 2016, instigated by Michael Atiyah, and organised by Michael Fourman, as part of the 2016 British Logic Colloquium http://conferences.inf.ed.ac.uk/blc/.

The panel members were: Michael Atiyah, Julian Bradfield, Alan Bundy, Mateja Jamnik, Ekaterina Komendantskaya, Maurizio Lenzerini, Phil Scott and Aaron Sloman.

After the panel discussion I read a collection of papers discussing aspects of mathematics, written over several years by Michael Atiyah, all included in his *Collected Works*, published by OUP namely:

Paper 147. Mathematics: Queen and Servant of the Sciences. Proc. Am. Philos. Soc. 137 (1993), No. 4, 527\u2013531. 521

Paper 150. Book Review in T.H.E.S of Conversations on Mind, Matter and Mathematics by Jean-Pierre Changeux and Alain Connes, Princeton University Press (1995).541

Paper 160. Mathematics in the 20th Century. Bull. London Math. Soc. 34 (2002), 1\u201315. 649

Paper 189. Mind, Matter and Mathematics, Berlin-Brandenburg Academy of Sciences, Berichte und Abhandlungen 14 (2008), p151--161. 269

Paper 191. The Athens Dialogues: Science & Ethics. The Spirit of Mathematics (unpublished) (2011). 291

I did not understand all the mathematical content, since my full-time mathematical education ended around 1959, though I have learnt various fragments since then. From some of the philosophical comments in his papers I gained the impression that at times Atiyah had shared the view of those who regard mathematics as a creation of human minds. (The "anthropological" view of mathematics proposed by Wittgenstein is discussed briefly below.)

But he is also aware that the environments of humans and other organisms have mathematical aspects independently of who or what is sensing and interacting with the environment. It is not generally noticed that such mathematical aspects can have a deep influence on a wide variety of evolutionary processes: making certain types of organism possible, and also influencing at least: types of reproduction, types of sensing, types of sensory interpretation mechanism, types of locomotion and other action, types of control, types of learning, types of reasoning, and types of communication -- all of which have mathematical aspects.

There may also be some mathematical constraints that are independent of the properties of the environment, e.g. constraints on possible forms of representation and reasoning that are independent of properties of the physical world, in the same way as the mathematical properties of the natural numbers (0, 1, 2, ...) are independent of properties of the physical world, but have probably been useful for several thousand years in many practical tasks involving physical objects and actions -- mainly because of their profound role as "standard" intermediaries in 1-1 correspondences. Later that role was extended to activities involving exchange and payment.

Moreover, some of the constraints on what is possible may be changed by products of evolution, e.g. producing new forms of nutrient, new types of "scaffolding" (explained below), and new forms of communication and collaboration -- all of which will have mathematical properties.

For example there are some mathematical properties of *forms of interaction* with the environment that depend in part on who or what is interacting, e.g. because different mathematical features of the environment are important for visual, tactile, haptic, auditory, olfactory, vestibular (inertial), electrical or magnetic sensing. These provide information with different mathematical forms because of the differences in what they are information about, and differences in how the information is received (e.g. contrast parallel visual projection of reflected light with serial haptic exploration of a surface). But a common physical reality with a rich mathematical structure underpins them all, cloaked by layers of biological and non-biological structure that can change over time.

More importantly, without rich pre-existing mathematical structure in the physical world biological evolution could not have happened and human minds and mathematical abilities would not have come into existence. As explained in the preamble above, that’s because biological evolution, and its products, depend on detailed mathematical aspects of the universe (different aspects as organisms become more sophisticated). I’ll say more about that below -- some of it echoing Schrödinger (1944), but going far beyond his concern with reproductive mechanisms.

Mathematical aspects of the universe and of evolution have contributed to major themes of the Meta-Morphogenesis project, from which this document emerged.

Is mathematics essentially human?

The ideas developed in this project go significantly beyond anything I have read about the nature of mathematics and its role in evolution. So it is likely that people who have not encountered or developed similar ideas will disagree with much in this document. I can’t tell whether Atiyah would disagree now even though some of his past writings are contradicted here.

However some philosophers of mathematics say things that I have implicitly contradicted above, especially those who regard mathematics as essentially a product of human thought, or human culture (e.g. Wittgenstein, quoted below). There have been more recent defenders of that sort of view, but I don’t have time to spell out in detail what they all get wrong!

Atiyah came close to endorsing that "anthropological" view of mathematics two decades ago:

> *"In describing mathematics as a language it is important to emphasize that a language is not merely a set of words and grammatical rules for producing coherent sentences. Words mean something and they relate to our experience. In a similar way a mathematical statement has a meaning and one which, I would argue, rests ultimately on our experience."*
> 150. Book Review in T.H.E.S of *Conversations on Mind, Matter and Mathematics*
> by Jean-Pierre Changeux and Alain Connes, Princeton University Press (1995).541

However, at the panel in Edinburgh he mentioned that his ideas had been changing, which may include no longer thinking of mathematics itself as a language, or as resting on human experience, as human mathematical knowledge does.

If "our experience" is somehow generalised to include all forms of practical use of information including non-humans selecting aims, making plans to achieve aims, understanding and reasoning about constraints, comparing alternatives, etc., then that brings the claim more in line with some of what I am saying.

In addition to kinds of mathematics relevant to users of information, there is also the kind of mathematical structure in the universe that makes it possible for such information-users (a) to exist at all, and (b) to be produced by natural selection.

I’ll try to offer an account of the metaphysics of mathematics showing that without the *prior* existence of a variety of mathematical structures with mathematical properties, biological phenomena, including human minds with mathematical and non-mathematical experiences, could not have come into existence.

So mathematics is not something created by humans, though mathematical features of the universe helped to make it possible for evolution to produce humans.

These were a mixture of features of the fundamental construction kit that made every part and aspect of the physical universe possible, including features of derived construction kits such as those required for producing vertebrates, the additional ones required for mammals, ... those required for producing a great deal of sensory and motor apparatus along with evolved construction kits enabling many kinds of information processing in virtual machines, some of which use several layers of meta-cognition required for mathematical thinking and other things. Much of that happened long before humans began to communicate linguistically.

Mechanisms required for human communication using vocal languages must have come long after other mechanisms required for (implicit) mathematical perception of and reasoning about spatial structures, relationships, possibilities for change, and constraints, since these are to some extent present in other organisms that do not appear to use human-like languages for communication -- or thought.

Observation of pre-verbal toddlers shows that they can apparently perceive and make use of topological relationships long before they can talk about them, as illustrated in this video of a 17.5 month child apparently creating and testing a hypothesis in 3-D topology about a continuous 3-D path between two different configurations of a pencil and a sheet of card with hole in it. http://www.cs.bham.ac.uk/research/projects/cogaff/misc/toddler-theorems.html#pencil

Mathematics is far more than a language, though many uses of mathematics, whether in humans, in machines or in other animals require uses of one or more appropriate languages, at least internally. This is partly analogous to the requirements for sophisticated computing systems to use complex internal languages, including in some cases interpreted languages rather than languages compiled into strings of bit patterns.

My ideas are partly inspired by Kant (1781), though he could not have thought of some of the details, given that he was writing around 250 years ago.

Part of the argument comes from Schrödinger (1944), though he discussed only mathematical requirements for reproductive machinery, not mathematical requirements for information processing in intelligent individuals able to sense and understand structures and possibilities in the environment, control many aspects of their behaviour, think, want, plan, learn, and in some cases communicate non-verbally.

Fragments of these abilities are shared with other species. They must all have emerged relatively late in evolution of humans and their precursors, requiring newly evolved internal forms of representation mechanisms for constructing and using them -- i.e. internal language competences whose primary function was not communication between individuals, but understanding and reasoning about possibilities and constraints in the environment. As suggested earlier, several further layers of meta-cognition had to be added to produce human abilities to think about their reasoning processes, discuss them with others, formulate generalisations, and organise acquired new information fragments and lines of reasoning into progressive structures shared with others. Some of that was a result of "collaboration" between biological evolution (changes in the genome) and cultural evolution, building on those changes.

The task of spelling out how all of those competences were products of detailed relationships between fundamental physics and chemistry, mathematics, requirements for biological evolution, and various evolved forms of intelligence, human and non-human, includes a particularly important subtask of *requirements analysis* that is closely analogous to the task of producing a requirements specification for a variety of increasingly complex information processing architectures to be implemented partly on physical machinery and partly using virtual machinery.

This is something we have come to understand only in the last 70 years or so (although Ada Lovelace anticipated some of the ideas in the previous century, as did Immanuel Kant even earlier). It is possible that there are still huge gaps in our understanding of these and other forms of information processing in humans that will hold back progress in AI for decades or longer, despite many and varied small advances in highly specialized task domains.

Part of the explanatory task is to develop, implement, test and debug instances of those designs, from which much would be learnt about inadequacies in the requirements specification, and how they can be improved!

Many different designs are needed because the requirements specifications for different sorts of organisms, including their different (implicit or explicit) uses of mathematics must cover a wide variety of types of organism, whose physical and information processing functions are extremely varied, including differences in their abilities to discover and use mathematical properties of their environments and properties of their own modes of sensing and acting in the environment -- very different for a crow, an elephant, a dolphin, and an octopus, to name a few unusually intelligent types of organism.

There may, however, be unnoticed overlaps between the mathematical requirements for control and use of the tentacles of an octopus, the trunk of an elephant and tongues in a wide variety of animals including humans and grazing mammals.

Unfortunately creating and testing designs for complex multi-functional virtual machines is not yet a standard part of the education of philosophers[Note on VMs], psychologists or neuroscientists, so most of them, including philosophers of mathematics, do not even understand the need for such a requirements analysis. Nevertheless, Kant managed to anticipate important aspects of that task.

Note on VMs
So the prediction I made in Sloman(1978) about how philosophy teaching would change in response to developments in AI, turned out false, unfortunately for philosophy.

Even if the claim that the meanings of our mathematical statements have something to do with our experience is true, there are also mathematical facts that are independent of our existence, and were necessary for the deep creative powers of biological evolution -- the most creative thing I know of -- long before humans existed.

Despite my seriously incomplete mathematical education, I shall try in the rest of this paper (and documents referenced on the web site) to construct a coherent (though still incomplete) account of the nature of mathematics, covering many aspects of mathematics, including its relationship with physics, life, evolution and computation, both natural and artificial, in the general sense of "semantic information processing".

NOTE
Those who believe, like the philosopher John Searle, that computation is concerned only with syntax, have little understanding of the processes of design, implementation, testing, and debugging complex working computational systems. Modern computational systems depend essentially on *referential* mechanisms, e.g. testing the truth or falsity of arbitrarily complex conditions in conditional branches. These conditions can have semantic contents referring to locations in memory, to external events, to mathematical relationships, and many more. Likewise instructions to be obeyed.

This general notion of information processing includes the roles of structured information in many biological and non-biological *control* processes, straddling a wide variety of time-scales. However, there are many gaps in the evidence that will have to be, for the time being, left open, or straddled by plausible conjectures, to be tested by their explanatory power until more direct tests become possible.

I have argued elsewhere against the view of information as essentially concerned with *communication*, since the most basic and most important role of information is in *control* -- a causal function, normally in combination with other causal factors that help to determine the consequences of control processes.

The communicative uses of information are derivative on its causal roles. Jane Austen (in *Pride and Prejudice* (1813) understood this over a century before Shannon's misleading terminology began to spread the unfortunate belief that information is inherently concerned with *messages* -- though he was not confused. His terminology was probably influenced by the fact that he worked for a communications company! (I owe that suggestion to Irun Cohen.)

For reasons that should become clear later, I'll focus mainly on the nature of mathematics and mathematical discovery before modern formal logic and so-called "Foundations" as developed in the 19th and 20th Centuries by Boole, Cantor, Church, Dedekind, Frege, Russell, Zermelo, Gödel, Turing, and others. However, the proposed explanatory hypotheses go back to a time long before there were any mathematicians, or even any life on Earth.

Although a great deal of influential early mathematics, e.g. much of Euclid's *Elements*, was concerned with continuous spaces, there was also discrete mathematics, including arithmetic. It is sometimes thought that discrete mathematics is recent (and there was a time when it was taught as a sort of oddity) whereas both the physical universe and requirements for even the simplest forms of life involve combinations of discrete and continuous structures and processes.

Part of the reason is that continuous spaces allow discrete changes, e.g. reversal or other changes of direction of motion, or motion from one side of a line or plane to the other or changes like coming into contact, coming apart, etc. From this point of view the need for discrete phenomena arise naturally as mathematical features of continuous phenomena. This goes against the "modern" view that the discrete natural number system is somehow fundamental and must be the basis of all other mathematical structures as shown by Dedekind and others. That does not seem to be how arithmetic and geometry were related in the minds of the great ancient mathematicians like Euclid and Archimedes. (That's a topic I have discussed elsewhere.)

Another way in which discrete mathematics structures and relationships exist independently of human minds was spelled out in Schrödinger (1944), showing how very stable, but changeable chemical structures with discrete states could be used to encode genetic information, partly anticipating some of Shannon's work and the later discovery of the structure and biological functions of DNA.

Moreover, in the physical world, where we perceive continuity, e.g. in fluids surrounding simple organisms, there are discrete molecular structures, and those chemical discontinuities are essential for life, in some cases making the difference between a nutrient and a poison. Detecting the difference can therefore be important even for very simple organisms. So although the study of discrete non-numerical structures (e.g. in logic and topology) came relatively late in human mathematics, they have been playing a central role in the functioning of life forms since the very earliest forms -- as spelled out in Ganti, 2003.

I have tried to show elsewhere [REF] that in AI/Robotics, psychology and neuroscience, a failure to understand the importance of partial orderings and discrete structures and processes, along with an excessive reliance on continuous measures, collection of statistics and probabilistic reasoning is seriously holding up both science and engineering.
[Part of the problem is that researchers are taught to use powerful programming tools that happen to support an inadequate variety of models of computation.]

Scope of this document

In the rest of this document I'll focus mainly on some of the relations between physics and mathematics, mathematics and biology (its roles in reproduction, evolution, and animal intelligence), and mathematical gaps in Artificial Intelligence (which may or may not depend on limitations of digital computers). Some of the main themes are

- Mathematics is not a human creation, though some aspects of human creativity involve use of mathematical relationships.

- Mathematical features of the universe existed long before there were any humans, and played a role in the evolutionary processes that produced humans and other products of biological evolution.

- Evolution and life depend on "construction kits", including the fundamental construction kit (FCK) provided by the physical universe before life began, and many derived construction kits (DCKs) produced later by physical and biological processes. The generative powers of these construction kits are consequences of their mathematical features. (A separate (long) paper, still under construction, is available: [*].)

- Many non-human animals, and pre-verbal human children, unwittingly make use of (partial) mathematical understanding of structures and processes in selecting and executing actions. More generally, the roles of mathematics in perception, reasoning, plan formation, practical problem solving, and learning -- not only in adult humans but also in other intelligent species, and pre-verbal humans ("toddler theorems") are not yet widely recognised or fully understood[*]. A consequence is that many important aspects of human and animal visual perception, including perception of processes, possibilities and impossibilities are still missing from AI vision systems. So current AI vision systems are far behind animal vision systems.

- Mathematics as it was known to ancient mathematicians about 2.5 thousand years ago, and taught by them, could only have arisen after basic animal mathematical competences were supplemented with *several layers of meta-cognition*, some of which involved cultural mechanisms.

- The discovery (by Descartes) that Euclidean geometry (or at least the best developed subset of it) could be accurately modelled in arithmetic and algebra was just one example among many of parts of mathematics that can model other parts. However that arithmetisation of geometry essentially "changed the subject". For example, it did not capture the features of Euclidean geometry that supported discovery processes that allowed Archimedes and others to notice the possibility of a simple, and natural extension of Euclidean geometry that makes trisection of an arbitrary angle easy: the "neusis" construction. (I am not saying that it is impossible to extend the arithmetical model of geometry to include a model of that construction.)

Some of the implications are discussed here:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/trisect.html

Frege's critique of Hilbert's axiomatization of Euclidean geometry was related to This. He did not notice that similar criticisms could be made of his attempts to reduce arithmetic to logic.

- Various attempts have been made to capture geometrical reasoning in AI theorem-proving technology. As far as I know the so far results are limited. In particular I don't think there is any AI system capable of discovering the neusis construction, or making most of the geometrical and topological discoveries of early mathematicians. I am not saying that that would be impossible. At least it shows limitations in our current understanding of the discovery processes and mechanisms involved in ancient mathematics.

- Although there is a great deal of mathematical content in Artificial Intelligence, and modern neuroscience, I suggest that theories developed so far fail to do justice to the nature of biological mathematical competences including competences of other intelligent species, and the kinds of mathematical discovery that were going on in ancient human mathematicians.

  I don't know whether this limitation of AI can be overcome by designing new types of virtual machinery and new types of information processing architecture, or whether it will require extending the physical forms of computation used in AI/Robotics, e.g. to replicate some of the functionality of chemistry in brains -- a requirement hinted at by Turing in his 1950 paper. (I'll say more about requirements later.)

- The great work leading up to Euclid's *Elements* and other ancient mathematics was done by superb mathematicians building on a wide range of advances in knowledge and understanding that must have come from a variety of sources, especially a large collection of human and animal competences required in practical activities.

  But the people who did that early "rational reconstruction" knew nothing about how to design information processing machines, such as intelligent robots. I suspect it would be very rewarding to assemble a multi-disciplinary team collecting evidence about missing details of the ancient discovery process and trying to build working models of the ancient mathematicians who made those discoveries, either using current computing technology (with a suitable collection of added virtual machine layers) or identifying precise requirements for extensions of digital computer technology that would make such modelling possible. (E.g. do brains make essential uses of chemistry, as hinted by Turing, in 1950, and what difference does that make?)

- During the late 19th and 20th centuries, human mathematical knowledge was dramatically extended and older portions partly reconstructed using a new framework (new "foundations") based on logic, set theory and formal methods. This is sometimes viewed as a process of discovering what mathematics (or some old portion of mathematics) "really" is, though I claim that's a mistake: it's a process of extending known mathematics with newly discovered sub-fields that include (partial) models of older parts of mathematics.

- The oldest discoveries, repeated in young pre-verbal humans (some of them shared with other species) do not depend on human languages used for communication. There may be overlaps with very much older languages that evolved for *internal* use in order to express contents of perception, intention, questions, wonderings, and other mental processes long before human languages for communication existed.

  (A claim that internal languages used to express contents that vary in complexity, vary in structure, and make use of compositional semantics evolved before languages used for communication, and develop earlier in children, is presented in
  http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#talk111
  Talk 111: Two Related Themes (intertwined).
  What are the functions of vision? How did human language evolve?

The place of mathematics in the universe

There's a long and complex story to be told, starting with fundamental physics (including mechanisms that make chemical structures and processes possible), and continuing through many stages of biological evolution, including evolution of ecosystems and cultural evolution.

Most such histories focus on changes of physical form, physical behaviour and environment, until they get to cultural evolution. That ignores important aspects of earlier products of evolution, including information-based control mechanisms of many sorts -- some used in reproduction, others in control of internal and external processes even in the simplest organisms. (Examples are given later.)

Information processing mechanisms and functions are required even in the very simplest organisms, to support control of nutrient intake, internal processes and physical behaviours, and also reproductive and developmental processes in which genetic information is transferred and used.

Implicit in the many varieties of information processing are uses of mathematics, including mathematical properties of the physical mechanisms, of space, time, and aspects of control. But requirements for types of control vary as environments inhabited vary, and also as needs and capabilities of individuals vary.

Most of these mathematical aspects of products of evolution are rarely noticed, in particular deep uses of mathematics by evolution and its products for many different purposes.

Asking these questions is in part a result of being influenced by Kant (1781) -- especially continually asking questions of the form "How is X *possible?"* or "Why *must* so and so be the case?" But we need to go into realms of which he knew nothing (e.g. information processing in virtual machines), though I think he would have welcomed them, and perhaps even contributed important ideas, had he lived two centuries later.

It's a pity that the phrase "Foundations of mathematics" has been captured for a different enterprise, I would have called this study of the earliest mathematical phenomena and their products, a contribution to the actual *Foundations of Mathematics.* The more modern topic should be described as "revisionary foundations" or "reconstructive foundations", as opposed to metaphysical and epistemological foundations.

Is mathematics an "anthropological phenomenon"?

The ideas I'll present below disagree with those who regard mathematics as intrinsically based on or connected with human minds and human languages. In particular, evolution made important use of mathematical facts long before humans (and human languages) existed. This conflicts with Wittgenstein's remark:

> "For mathematics is after all an anthropological phenomenon."
> *(Remarks on the Foundations of Mathematics)*

which ignores all these points about the pre-history of humans.

Of course, there is a related tautologous claim not about mathematics, but about human mathematical knowledge -- it is, after all, human!

But mathematics is more than human mathematics and what makes its contents true is not human decisions or preferences. Many of the uses of mathematical information by humans are just special cases of the use of mathematics in products of evolution. Other special cases include many superb control functions in non-human animals, e.g. nest building birds, birds of prey, hunting mammals, and expert movements through tree-tops by various monkeys and apes, and squirrels!

Atiyah does not explicitly endorse Wittgenstein's view in the papers I have read, although some of what he says about human mathematical knowledge looks superficially similar:

> *"There is little doubt that mathematics has emerged from our experience of the natural world either visually in geometry or discretely by enumeration and counting. But mankind has done more than just observe, it has built internal concepts and structures that reflect what it has seen. Analogy and abstraction are the key ideas involved, and they remain as pertinent now as two thousand years ago. The whole impressive edifice of mathematics is a human creation built on observation of the natural world."*
> The Athens Dialogues: Science & Ethics. The Spirit of Mathematics (unpublished) (2011). 291
> http://www.panoreon.gr/files/items/1/163/the_spirit_of_mathematics.pdf

That presents aspects of a subset of mathematics -- as a product of human minds. But I don't think Atiyah believes there is nothing deeper, more general, and independent of the existence of humans -- e.g. mathematical features of the natural world that exist whether we observe them or not.

In particular, there are mathematical facts about the physical universe (including facts about space and time), and also mathematical facts about biological mechanisms, about ancient products of biological evolution, and about intelligent non-human species that often go unnoticed. When they are noticed their mathematical aspects are not always identified. Likewise some "cute" but unconsciously mathematically sophisticated behaviours of pre-verbal human toddlers.

There are also much older biological functions that require problems to be solved using information -- and these have mathematical features, e.g. the use of negative feedback in many biological control systems -- homeostatic mechanisms. Moreover, one of the main points made by Schrödinger (1944) is that mathematical aspects of chemical structures producing multi-stability have mathematical consequences in long molecular chains that are essential both for reliable reproduction and also for supporting evolved diversity through informational diversity in genetic information.

All of that arises out of the mathematical fact that each time a specification is extended by a fragment for which there are two options, the total set of possibilities is doubled (or multiplied N-fold for N options per item). This leads to exponential growth of expressive power.

There must be many still unknown mathematical features of the fundamental construction it and derived construction kits that may eventually help us understand how mechanisms in human brains (still unknown) together with extensions produced by human cultures, made it possible for humans to produce/discover think about, teach, learn and use aspects of mathematics that originally grew out of mathematical competences of non-human ancestors.

These ideas generate questions such as:

> *What sorts of mathematical functionality were being used by other species before evolution produced humans?*
> *How do specifically human mathematical abilities, e.g. spatial reasoning abilities, relate to those of other species and to the structure of the universe in which we evolved and live?*
> *And how do they develop in individuals -- human and non-human?*

I'll suggest some partial answers later. Jackie Chappell and I have some ideas about epigenesis that as far as I know are not widely shared. Chappell & Sloman (2007)

Interlude: The educational importance of Meccano

My thinking about the nature of mathematics is deeply influenced by experience of playing with a succession of increasingly complex Meccano sets between the ages of 5 and about 10, followed by learning Euclidean geometry at school, using a textbook with many problems at the end of each chapter ("Prove that...", "Find a construction that ...", etc.).

> **NOTE:**
> Euclidean geometry seems no longer to be a standard part of mathematical education: a tragic failing of recent education. I've noticed some of the consequences in AI/Robotics researchers, and also psychologists and neuroscientists discussing mathematical cognition, and philosophers discussing philosophy of mathematics.

As remarked above, the basic components of a construction kit allow elementary transitions, e.g. combining and separating parts, or altering relationships of combined parts.

In some cases the changes are all discrete -- e.g. the basic operations on standard lego blocks -- whereas other kits, like meccano and tinkertoys allow continuous variation during and after construction.

It is sometimes forgotten that although developing continuity out of discreteness is very difficult (perhaps first specified mathematically by Dedekind?) the reverse is much easier.

For example, if something is moving along a line and switches to moving in the opposite direction this can be detected as a discontinuity even if space and time are continuous in some intuitive sense.

Likewise the continuity of a plane is disrupted by a line (visible or invisible) in the plane: points moving from one side of the line to the other undergo a discontinuity in region membership even if their motion is continuous.

There are also many ratchet-based devices that produce discontinuities and irreversible transitions out of continuous reversible processes (with the aid of a spring). Many biological mechanisms produce discontinuities, e.g. teeth coming into contact or moving apart, or seeds with hooks that catch on to wool or fur of passing animals and get pulled off the original plant.

Hands with fingers can produce discontinuous motion of other objects by pushing fingers through holes and then moving sideways, or using them as hooks.

There are huge numbers of organisms with mouths, teeth, tongues, claws, fingers, stings and other physical mechanisms that can produce discontinuous changes in spatial and causal relationships through continuous spatial movement. Vision also allows continuous motion of the perceiver to produce discontinuities in what is visible in the environment, e.g. moving an obstacle can suddenly produce or block a view of a distant object, and a perceiver's change of viewpoint, or rotation of an object can change what is visible on surfaces of objects in the environment -- essentially because visual information travels in straight lines and the lines can be blocked or unblocked in various ways.

The facts I have mentioned, and many more (including many not known to me) imply that long before humans existed on this planet there were many opportunities for other more or less intelligent species to evolve ways of acquiring, processing, storing and using information about static and changing spatial information, with continuous and discrete structures and relationships (e.g. betweenness).

Shallow, inefficient and unreliable mechanisms would merely learn statistical relationships between sensor values and motor signals. A much deeper, richer and more powerful type of information processing could be based on evolved or learnt mechanisms for representing spatial structures, relationships and processes both in contact with parts of the organism and existing at various distances, and in some cases continuing to exist while not perceived -- providing opportunities for using changing geometric and topological relationships between the perceiver, perceived objects and unperceived objects to plan and predict changes -- instead of being stuck with the task of discerning useful correlations in sensory and motor input and output signals.

Such statistical learning could never achieve knowledge of impossibilities and necessities or even spaces of possibility, as opposed to mere samples of spaces.

Use of semi-metrical *exosomatic* information (networks of partial orderings about things and processes in space) relating to contents of the environment outside the organism not *somatic* information about contents of input and output signals, would allow various forms of evolved intelligence to be used to acquire and use information about objects and locations in the environment including enduring objects and locations that are out of sight -- long before the invention of maps based on uniform coordinate systems.

This assumes that the fundamental and derived construction kits (and scaffolding) available to evolution included information processing apparatus capable of being used to refer beyond an organism's skin. I once suggested that this could be achieved by extrapolating the functionality of mechanisms that originally evolved for reference to internal parts, states, and processes of an organism, Sloman(1985) and Sloman(1987), though I don't think the idea has ever been developed in detail. Intriguing special cases were demonstrated by Kuipers and Modayil (REF) using a robot with a one-dimensional retina sensing the static and changing contents of a 2-D world including remote objects reflecting laser beams.

What we now know as Euclidean geometry could have been developed by adding more constraints to such an extrapolated spatial ontology, along with more precise forms of representation, more complex chains of reasoning, and richer memories for constructing and organising spatial information across extended time periods.

These changes almost certainly required many genetic discontinuities adding both new sensors, new means of controlling sensors, new means of transforming and combining information, new means of controlling motors over longer time intervals (so that planning for actions in future states is needed).

It is also likely that different lineages evolved comparable or overlapping information processing capabilities, but with many differences in sensory details, representational mechanisms and motor control mechanisms. Differences between intelligent birds and intelligent mammals with functionally equivalent capabilities (e.g. finding, opening and eating nuts) may perhaps shed light on some of the evolutionary processes -- including perhaps the possibility of similar evolved virtual machinery with similar mathematical functions, implemented on different physiological mechanisms: convergent evolution based on the generative potential of a common FCK and perhaps some shared DCKs.

All this implies that there is vast potential for evolution to discover many varieties of continuous motion and continuous and discrete types of state change and their relationships.

It is likely that many these processes evolved (e.g. in birds, and some mammalian and reptilian species) long before humans evolved, and humans, for some reason extended what had evolved before them with much richer meta-cognitive mechanisms and capabilities, that eventually were linked to richer forms of communication and joint action, leading to communication and debate about which forms of reasoning worked when, and why, eventually giving birth to Euclid's Elements and other achievements of ancient mathematicians. [That's much too compressed.]

A multi-disciplinary project to investigate all this could be exciting, except that our educational system and university research-training regimes do not produce the required variety of interests and competences, especially when so much funding and academic recognition depends on relatively short term economically useful results.

Moreover, it is not clear whether current computing technology could in principle suffice (supporting new kinds of virtual machine architecture) or whether something very different needs to be developed that currently is available only in the brains of intelligent animals. If so, discovering what those mechanisms are and finding ways to replicate them may be well within reach, or may require scientific and engineering advances that are decades away, or longer.

The rest of this is even more of a mess, and perhaps should not be read until a new version is installed with this comment removed. Suggestions and criticisms welcome anyway.

New section???

I suspect that similar facts are used in some of the oldest biological control systems, using information to trigger state changes or actions. Sensors that are very poor at measuring absolute values of some physical or chemical state accurately may be very good at detecting change of direction of change.

They may also be good at detecting states that pass some threshold, if that threshold corresponds to a discontinuity in a physical detector, e.g. a sensor state overtaking/passing another sensor state. Schrödinger (1944) points out that quantum mechanisms allow discrete state changes to be used in biological organisms. However, our previous remarks show that even if no physical mechanisms are inherently discrete changing mechanisms combined with comparators can detect and use what might be called "virtual discreteness".

There are different ways of embedding discrete changes in continuously changing systems that are important in chemistry -- the use of catalysts. (This point needs to be expanded.)

I suspect that many of the obstacles to progress in designing intelligent machines, including machines with animal-like visual capabilities, arise from the engineering commitment to measure continuously varying values with inherently noisy and inaccurate sensors, and then attempting to deal with the noise and inaccuracy using statistical information and probabilistic reasoning. I suspect that switching to attempting to make far more use of partial orderings and various kinds of induced or inferred discontinuities would be a game-changer. But the required forms of mathematical (and philosophical thinking) don't seem to be available in current graduates. (It's not something I could do on my own.)

Progress may require us to acquire a deeper understanding of the Fundamental construction kit provided by physics and chemistry are also important in human designed construction kits including the continuous/discrete distinction mentioned by Schrödinger.

Building meccano models can lead to discovery and understanding of changing possibilities and constraints as the complexity of a partially built meccano model increases. This seems to me to be closely related to processes leading up to discoveries reported in Euclid's Elements.

An obvious example is the relation between rigidity of a triangle in meccano and the Euclidean theorem that having three sides equal makes two triangles congruent.

The relationship between intuitively understanding the former and having a Euclidean proof is complex, and unclear. (I have some rough ideas.)

Other examples of relations between meccano and Euclid are correspondences between theorems about the "locus" of a Euclidean point satisfying certain constraints (e.g. moving in such a way that its distance from a fixed point is constant, or distances from two other points remain equal), and the types of constrained motion that can be produced by particular meccano constructions.

Likewise there are topological relationships constraining possible motions, e.g. the possible motions of a piece of string after one end has been passed through a hole in a rigid plate and joined to the other end. A less obvious example is getting a spoon into position to stir a mug of liquid: this requires understanding (at least roughly) which portions of space above the level of the mug are part of its cylindrical vertical extension, working out a trajectory to get the spoon into that region of space and then moving it downwards. Moving it vertically down from locations in other spacial volumes cannot produce the desired result. The process will also fail if the spoon is held horizontally with portions projecting out beyond the (virtual) cylinder.

An astute learner might discover the transitivity of 1-1 correspondences through meccano experiments: initially as an empirical discovery, and then later as a deep structural necessity.

Many more examples can be observed in birds constructing nests, e.g. crows and weaver birds, squirrels defeating "squirrel-proof" bird feeders and pre-verbal human toddlers[*].

Perception and use of these structural relationships (in the environment, not in spaces of configurations of sensorymotor signals) and their implications remains a serious problem for AI/Robotic vision systems. Many of the researchers don't even understand the problems.

Such practical applications of intuitive mathematical understanding of what is and is not possible cannot be achieved by collecting statistical data from many repeated experiments, as Kant understood well. Statistical data may suggest probabilities but cannot establish impossibilities or necessary connections. This is a big gap in current AI, which has barely been noticed by AI researchers.

It's also a serious gap in what neuroscientists can explain or model. At present I don't think anyone has the faintest idea of how brain mechanisms enabled the great mathematical discoveries of Zeno, Euclid, Pythagoras, Archimedes, and other ancients, or which sorts of neural development around the age of 5 or 6 years makes a child able to understand that one-one correspondence between sets is necessarily a transitive relation [Piaget 1952].

Mathematical properties of ancient naturally occurring construction kits

Our pre-Euclidean ancestors did not play with meccano, but I am sure that their natural environment and activities -- such as building shelters and bridges, making tools and weapons, climbing complex structures such as trees and rock-faces, and probably also dismembering fruit, nuts and captured prey in order to get at the edible portions inside-- also contributed to an important collection of proto-mathematical cognitive competences concerning geometry, topology and what used to be called "mechanics".

I am not claiming that those ancestors all had the meta-cognitive ability to notice what they were learning, or what assumptions they were making. That came much later, after new brain mechanisms had evolved, with consequences discussed below.

Example: if you hold a rigid stick horizontally at one end and rotate your hand in a horizontal plane the far end will move in a horizontal plane in a curved line, in a direction determined by the direction of rotation.

Example: a stick held vertically that is too long to go through a horizontal slit in a wall, can be rotated in a vertical plane parallel to the wall until it is parallel to the slit. It may then go through the slit. If it is still too long it an be rotated in a horizontal plane until it is perpendicular to the slit. It can then go through the slit if its diameter is less than the width of the slit.

There are many more examples of mathematical consequences of motions of rigid objects and non-rigid but unstretchable objects (strings). Intelligent animals, and pre-verbal human toddlers understand many of these, though their understanding is not perfect (as Piaget, Lakatos and others have shown).

When such things are understood they don't need to be tested in different locations, with sticks or walls made of different materials with holes of different lengths, etc., as would be necessary if the discoveries were merely statistical generalisations from many observations.

Many researchers seem to think all learning is statistical: they ignore the many mathematical discoveries intelligent animals can make.

One of Kant's examples was the ability to discover that information about the sequence of observations of features of a building as you go round it in one direction can *necessarily* be used to infer the sequence you would observe going round it in the opposite direction if the building has a fixed structure. This requires abilities to represent and reason about ordering relationships.

There is a rich class of proto-mathematical competences concerned with acquisition of visual information.

For example, implicit understanding that visual information (normally) travels in straight lines can be used in deciding how to change one's location in order to get new information, e.g. getting information about invisible (rear) parts of a large visible object.

Likewise understanding how to move in order to see new portions of a large space visible through a restricted opening.

Closely related is the ability to rotate an object to get more information about it, or how to move one object to get more information about another that is partly occulted.

Related points can be made about knowing which way to move a hand or paw to get new haptic or tactile information about surfaces. Human babies use their mouths for exploring many shapes, though it is not easy to observe what's going on in their mouths.

So, not only manufactured construction kits, but also many naturally occurring objects provide opportunities for playful exploration that can lead to mathematical discoveries of consequences of various combinations of changes made to increasingly complex combinations of objects.

Evolution also makes useful mathematical discoveries by "playing"

The physical/chemical materials on this planet before life formed provided a sort of construction kit, built from the FCK, that must have been sufficient somehow to produce the earliest life forms. In doing that, matter was "blindly playing with itself" and "blindly making mathematical discoveries about itself".

The discoveries were implicitly expressed in both (a) functioning organisms and (b) the structures that controlled the formation (and therefore reproduction) of those organisms.

Insofar as the successful functioning and reproduction of each type of organism depended on structural relationships between (a) the physical designs of the organism and (b) the information about how to create new instances of the designs, where both (a) and (b) made use (implicitly) of mathematical relationships between structures and processes, this can be seen as a very early biological use of mathematical and meta-mathematical information, billions of years before humans began to build theories about either.

The enormous power and creativity of evolution depended on both (a) and (b) becoming parametrised in ways that supported systematic variability

-- (a) supporting changes of behaviour and function of a single organism at different times in different places,

-- (b) supporting changes across developmental stages of individuals and across generations, then across species.

Without this generative (mathematical) power in the construction kits, including mechanisms (physics and chemistry) and forms of representation (computational specifications), Darwinian natural selection would not have been able to occupy such diverse niches. Many admirers of Darwin seem to forget that *selection* mechanisms cannot produce or explain successful diversity unless there are also *generative* mechanisms.

Consequently they don’t think about the features of the generative mechanisms and the mathematical requirements for them to be able to support the enormously varied products of evolution already on this planet.

I suspect that deep new investigations of the mathematical features of early bootstrapping processes in evolution will reveal previously unnoticed mathematical features required to support not only formation of physical structures, but also information processing capabilities, in both the fundamental and early derived construction kits.

These requirements, in turn, could impose new constraints on acceptable fundamental physical theories, possibly leading to new types of mathematics if existing types don’t suffice for reasons hinted at earlier (e.g. the need for something more like generative grammars than like sets of equations.)

What sorts of functionality? Schrödinger suggested that ordered sets of molecules with switchable states could encode genetic information satisfying requirements for huge amounts of variability (extendable over time simply by lengthening molecular sequences).

But, as far as I know, he did not attempt to specify the sorts of machinery that could make use of that kind of structural complexity and variability. He may have known about Turing machines, but did not mention them.

Once types of organism with the minimal features to support reproduction based on copying had emerged (perhaps with the features described by Ganti), natural selection could provide a new kind of blind exploratory play, leading to new (blind) mathematical discoveries, subsequently used (blindly) in the production of increasingly complex and increasingly varied forms of life, with increasingly complex mathematical properties. At later stages some of the uses of evolutionary affordances are not blind, e.g. when mate selection occurs in intelligent organisms, or selective breeding is used.

Pre-human mathematics

Wittgenstein was deeply mistaken because mathematics is a multi-faceted phenomenon, with several non-anthropological aspects, on which I have tried to elaborate, based on the idea of construction kits and their products. There are many types of construction kit with different mathematical properties related to their generative powers.

Many biological uses of mathematics (e.g. in control mechanisms, including control of reproduction, control of development, control of learning, and control of individual actions) long preceded the "anthropological phenomena", as indicated in the partial history below.

In particular, biological uses of mathematical properties of structures, processes and control mechanisms started long before humans existed.

Many (perhaps all?) varieties of mathematics are concerned with properties of concrete or abstract "construction kits" with generative powers. (Modern research on so-called "foundations" restricts the types of construction kit in narrow ways, to which I'll return later.)

Some construction kits are concrete because they include physical matter, like meccano, lego, tinkertoys and plasticine, whereas other construction kits are abstract, e.g. grammars specifying linguistic or non-linguistic structures, and games like chess that can be played using a physical board but need not be. Other abstract construction kits are parts of abstract algebra, logic and computer science.

The Fundamental Construction Kit (FCK), from which all other concrete construction kits are derived, must have existed throughout the life of the universe. (Could it change over time, or be modified in different ways in different parts of the universe?)

The FCK must have deep mathematical properties -- not all of them understood yet, as discussed later.

New derived construction kits (DCKs) seem to be constantly under construction, often by processes of biological evolution, but also in older processes, including geological processes and effects of asteroids, that helped to produce the conditions for biological evolution on this planet, and more recently in the products of human engineering and art (hardware and software).

New DCKs can have new mathematical structures and capabilities, which, in turn, can extend the challenges (especially information processing challenges) of organisms in the same environment. Camouflage is one of the very old examples.

Towards a theory of construction kits (With mathematical properties)

For the last few years, as a side effect of the Turing centenary, and especially writing a commentary on Turing's paper 1952 paper "The chemical basis of morphogenesis"), I have been trying to develop a theory of construction kits and their roles in biological evolution. Turing had described a kind of chemical construction kit, and explored some of its mathematical properties. I take that to be a very simple special case of a very important, deep, long term project: the

Meta-Morphogenesis project which includes a study of how some of the evolutionary side-effects of mechanisms of morphogenesis produce modified or new forms of morphogenesis.

Many of the ideas are still quarter/half-baked, partly because the project requires deep collaboration between researchers in physics, chemistry, biology (including psychology and neuroscience), CS/AI, philosophy, and mathematics. But as it’s directly relevant to "What is mathematics?" I’ll summarise some themes.

The earliest structures and processes with mathematical properties are parts or aspects of a Fundamental Construction Kit (FCK), comprising aspects of the universe that must have been there at the start, or must have always existed if there was no start.

For every *concrete* construction kit (CCK) there is at least one corresponding *abstract* construction kit (ACK) whose properties mirror major features of CCK, in the way that diagrams involving parallelograms mirror features of composition of forces, or Newton’s laws mirror features of matter in motion. (There will also be many related ACKs with variations.)

As with construction kits like Meccano, etc., each CCK and each ACK has basic components and modes of composition producing new components. The new more complex components will have *derived* features, including new possibilities for change -- adding or removing new, more complex, parts, removing parts, or altering relations between existing parts.

There are many such Derived Construction Kits (DCKs) produced directly or indirectly by evolution, though they are not necessarily all parts of the organisms they help to construct and/or maintain.

Evolution can be opportunistic: organisms can make use of something new that happens to become available as a side effect of their own actions or actions of other species, or physical events and episodes such as asteroid impacts or volcanic eruptions or ice-ages.

Construction kits typically have the potential to produce many different sorts of product; but not all sorts can be simultaneously realised, so the histories of construction kits and their actual products (including derived construction kits discussed below) are normally a tiny sample from the space of possible trajectories.

Unlike some many-worlds theories according to which different possible worlds correspond to different assignments of values to a *fixed* collection of variables, the different possible products of construction kits can introduce new types of complexity, requiring increasingly complex specifications.

This can be compared with the way in which a recursive grammar can generate arbitrarily long sentences with a huge, or infinite variety of grammatical structures. For physical/chemical structures that sort of potential may be limited by the size of the universe, or by gravitational forces in very large structures. Nevertheless this analogy suggests that the mathematics required for describing the possible structures in the physical world is unlike the kinds of mathematics normally encountered in physics. It’s more like what we find in chemistry, or linguistics.

I suspect the mathematical study of varieties chemical structures and processes formulae is more complex, and less well developed than the mathematical study of syntactic structures and processes (e.g. generation and parsing processes).

This has been fairly obvious for chemical structures and processes for some time and there are many computational models of chemical processes that are very different in character from the formalisms normally used in physics.

But I don’t know if anyone has a systematic theory of the space of possible chemical structures and processes: it would be enormous.

So, rather than a system of equations with fixed complexity (e.g. a fixed number of variables) , the increasingly complex possible products of a construction kit are more like a tree-structured, or network-structured branching space of nodes with varying complexity, including possible new types of organism or possible new components of organisms, or possible new competences of organisms.

There will also be a space of possible processes involving all of those.

Compare products of a recursive grammar -- allowing us to produce sentences and paragraphs that have never previously been constructed or understood although their smaller parts have, including, I suspect, this sentence!

> **NOTE**
> If every physical structure were composed entirely of point masses each with a number of coordinates and derivatives, etc., as in Newtonian physics, then chemistry as we know it would have been impossible. (I think there is some evidence that Newton grasped this point, which was perhaps implicitly assumed by alchemists?)

Every such structural possibility when realised produces further sets of possibilities, i.e. new extensions or modifications of the newly realised structure (unless nothing else can ever combine with it).

A new structure can also create new IMpossibilities, since old possibilities can be removed by new constraints resulting from addition of new parts and relationships (e.g. adding a new strip linking two parts of a meccano model can eliminate some of the previously possible movements).

A more subtle example, is the fact that 12 similar sized cubes can be arranged in an NxM array where N and M are both integers > 1, in more than one way, whereas if one cube is removed, or a new one added, the new total of 11 or 13 cubes can no longer be arranged in such an array.

(A child may discover this without being sure that the rearrangements really are impossible, because the child is not able to link this to the unique factorization of positive integers and the existence of prime numbers.

A really clever child might find a proof of impossibility by superimposing the grid on a regular array, and exhaustively exploring all possible rearrangements within a large enough square, though in some cases proving the impossibility of an NxM array by exhaustive search could take far more than a lifetime.)

Such reduction of degrees of freedom in a small structure can be a step towards a more reliable foundation for something new that is more complex, with yet more degrees of freedom: e.g. building the base of a tower out of larger, stronger, more rigid components, so that it can support a larger, heavier structure reliably. The supported structure may be flexible, e.g. a large crane, or digger.

The previous paragraphs give a brief and patchy introduction to some ideas about construction kits and their mathematical properties. If the universe has a fundamental construction kit as suggested, its mathematical properties will have nothing intrinsic to do with humans, human languages, or human thought, except insofar as its generative potential will include a vast number of possible physical structures and processes, including evolution of human beings.

NOTE
It is possible that the ideas about layers of new kinds of construction kit, including construction kits for building information processing systems using virtual machines, can provide answers to the deep questions, e.g. about how scale changes can produce fundamental changes, posed in Philip Anderson's famous 1972 paper, whose semi-final paragraph stated:
"Surely there are more levels of organization between human ethology and DNA than there are between DNA and quantum electrodynamics, and each level can require a whole new conceptual structure."

It's only in the last 70 years or so that we have discovered how to design, implement, test, debug, extend and use machines like that. But understanding of what we have achieved thereby is still very shallow, and not very widespread among those who need to know. And there are still kinds of VMs we don't know to build, such as those used in the minds of ancient mathematicians exploring Euclidean space, including discovering the construction that makes trisection of an arbitrary angle easy, mentioned above.

Concrete vs abstract changes

Realisation of possible changes can have different implications in concrete construction kits (e.g. adding a new piece to a meccano construction) and in abstract construction kits (e.g. moving a piece in a game of chess, or distributing an operator over an algebraic expression, or modifying a rule in a game, or an axiom in a logical system).

There can also be structurally similar mathematical consequences of abstract and concrete changes: including alteration of remaining immediate possibilities for change.

Biological evolution on this planet would have been impossible but for the concrete construction kit provided by physics and chemistry: the Fundamental Construction Kit (FCK), whose properties are not yet fully understood.

All other concrete construction kits are derived from the FCK: they are derived construction kits (DCKs), and some of them introduce new possible structures with new mathematical features that can be reached in far fewer steps than would be required using the FCK -- a fact that is used repeatedly by advances in engineering.

Similar changes in sets of directly reachable possibilities (and related constraints) were produced by natural selection long before human engineers produced new engineering short-cuts, as we'll see.

So there were lots of structures and processes with mathematical properties and constraints and generative potential, but nobody to notice them, describe them, or derive their implications.

But a tiny subset started *implicitly* noticing them, using them and deriving implications -- after the FCK had produced early life forms and an environment in which natural selection could occur.

Natural selection produced many structures and processes derived from the FCK. In doing that it implicitly proved some theorems about what is possible for the FCK.

That’s how mathematics got going long before there were any humans. I’ll return to human mathematics later: there’s a long story in between.

Schrödinger’s ideas

On my reading of Schrödinger’s little book "What is life?" (1944) he was drawing attention to some of the surprising mathematical features of the FCK that are relevant especially to encoding information needed for reproduction.

Quantum mechanics is strongly associated with indeterminism, in contrast with Newtonian physics. Despite that, QM also provides what at first seems to be the opposite of indeterminism namely kinds of structural stability that are not possible in a universe composed only of Newtonian particles.

A highly stable configuration of atoms can be transformed, by a large enough packet of energy of the right sort, into a different highly stable configuration (he illustrates this with the possibility of relocating the oxygen atom in a propyl alcohol molecule to produce a different isomere, with different chemical properties).

Some of those discrete changes between stable structures require a relatively large packet of energy, whereas others make use of catalytic mechanisms -- partly analogous to the action of keys on locks.

That is another important (mathematical) feature of the FCK used in essential ways in all life forms, including the simplest, as explained by Tibor Ganti in his book "The principles of life" usefully reviewed by Gert Korthof here:
  http://wasdarwinwrong.com/korthof66.htm
  (OUP should have provided a cheap edition by now.)

Schrödinger showed that QM’s (superficially contradictory) combination of both radical indeterminism and very high degrees of stability (in certain ranges of temperatures, etc.) in complex molecules is essential for many aspects of biology, including the use of complex molecular chains for stably encoding reproductive information.

It is also essential for the formation and use of new highly functional complex molecules in the growth and functioning of individual organisms.

That combination is also essential for some of the new physical structures that produce new opportunities and threats facing previously evolved species.

[I have made some of Schrödinger’s text, with comments, easily available here:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/schrodinger-life.html]

In Section 34 he wrote this, emphasising a newly discovered mathematical property of the universe (a property of the FCK):

> *"The great revelation of quantum theory was that features of discreteness were discovered in the Book of Nature, in a context in which anything other than continuity seemed to be absurd according to the views held until then."*

(Note that he is referring to mathematical features of the universe that do not depend on the existence of humans, and could not properly described as in any sense "anthropological", unlike their discovery and discussion by humans.)

I.e. for some combinations of atoms (and sub-atomic structures) there are discrete sets of (relatively) stable states, with mathematically distinct structures despite common features such as total mass, total number of sub-atomic particles, total number of bonds, etc.

Another mathematical feature: Continuous change

In addition, many complex molecules, especially molecules in reproductive mechanisms are *also* subject to continuous changes such as folding, twisting, and moving together or apart. Such processes are essential for construction and functioning of biological organisms.

So discreteness and continuity are both *mathematical* properties of the FCK and its biological products.

And both are essential for biological reproduction, and in some cases also biological evolution in this universe.

That combination of discreteness and continuity, may be more important than generally realised, for instance if it supports a richer variety of types of control and information processing than previously thought. (I suspect Turing was exploring such ideas shortly before he died.)

Meccano assemblies, unlike basic Lego brick assemblies, also have a combination of discrete and continuous sets of possibilities and impossibilities (constraints). (This ultimately depends on a similar fact about the FCK.)

Compare plasticine, which supports mostly continuous change, except for topological discontinuities such as separation, joining, forming rings, etc.

In all those kits, including the ones supporting discrete sets of possible structures, the *construction processes* required involve continuous spatial changes e.g. bringing a new lego brick into the right position to be added to extend a rigid lego structure.

(Ron Chrisley pointed out to me that one can also ’cheat’ with lego bricks and produce a continuously variable hinge -- presumably an unintended consequence of the design. Another cheat(?) would be placing a disconnected lego brick inside a larger otherwise empty space in a lego construction.)

Logical and algebraic formalisms and most computing formalisms form construction kits with only discrete sets of possibilities, although they can to some extent model continuous changes.

It’s not yet clear what the full implications of that restriction to discrete changes are, though many continuous processes can be approximated discretely to varying degrees with varying costs.

There may be some serious physical limitation to continuously branching sets of possibilities, even if discrete branching (e.g. binary forking) can go on indefinitely.

Moreover, the exponential growth involved may hit obstacles at a certain size.

I suspect physicists still lack knowledge of some of the deeper mathematical features of the physical, chemical, spatio-temporal universe on which biological evolution depends.

(That may in part be a consequence of using a restricted set of mathematical formalisms that limit the space of theories that can be constructed. E.g. contrast the use of vector spaces with use of grammars.)

Evolution as a blind mathematician

I am claiming that long before there were any human mathematicians biological evolution, and its products, blindly/unwittingly made essential use of mathematical properties of the fundamental construction kit.

As products of evolution became more complex and more varied it made different kinds of use of mathematical properties of both the FCK and derived construction kits (DCKs), e.g. some uses involve *creating* and *modifying* physical structures and processes, while others involve construction kits for controlling behaviours by programming.

In some cases evolution makes temporary use of structures produced by earlier evolution and some found in the physical environment that are not parts of or products of the new organisms they help to create.

I call those temporary aids "scaffolds". Supports used by climbing plants are examples. Sometimes one species uses another as scaffolding, e.g. parasites.

Sometimes there is mutual scaffolding, e.g. in symbiotic relationships.

In mammals a placenta is part of the scaffolding of a new infant, created by the mother then discarded. Eggshells and nutrient stores in eggs and seeds can also be regarded as scaffolding.

As with components of organisms, the mathematical properties of scaffolds affect their functions -- including supplying nutrients and disposing of waste in the case of the umbilical cord.

The mathematical properties of scaffolds can be as important for the species that use them as the mathematical properties of their own bodies, genetic mechanisms, etc.

Compare the differences between living suspended in water, and being able to move around on solid relatively stable, multi-faceted structures, with reliable sources of food, shelter, etc. in different locations.

The physical and mathematical properties of such extended richly structured inhabitable spaces offer not only opportunities for evolving new physical mechanisms for locomotion, but also opportunities for evolving new useful forms of information processing -- including abilities to synthesise fragments of information gained in different places at different times to form relatively stable information frameworks within which changes in space occupancy can be represented. (E.g. the earliest "cognitive maps").

Details will vary enormously across organisms with different needs, different physical structures, different information processing capabilities, and different possible modes of locomotion (including slithering, crawling, digging into soft surfaces, walking, running, jumping, climbing, and flying).

J.J. Gibson noticed some of the implications for perceptual information processing, including, detecting optical flow and changes in texture gradients.

His theory of perception of positive and negative affordances as possibilities and obstacles for action for a perceiver is too limited. It needs to be extended to include perception of possibilities for relative motion and constraints on motion that involve other things than the perceiver.

An example would be seeing that a container is too wide to go through a doorway, but if rotated so that its longest diameter is held vertical, then it can go through the doorway.

Gibson[1979] discussed only a subset of cases involving mathematical structures such as texture gradients and optical flow patterns on a retina produced by relative motion.

I don’t know whether he was aware that he was sampling a tiny fragment of a huge space. As far as I know he did not notice the connection between the phenomena he discussed and mathematical discovery processes. He was mainly thinking of competences relevant to a variety of species that use vision to control motion, ignoring the role of vision in human mathematical discovery.

In particular, unlike Piaget [Possibility/Necessity], Gibson and most other vision researchers failed to notice that intelligent organisms need to be able not only to perceive actual structures in particular locations and particular processes occurring when those structures move, but also to perceive and reason about *possible* processes that are able to change relations between structures or produce new relations, e.g. more stable support, and also constraints or necessary consequences of changes, that make some processes or some end states impossible (as Kant observed).

It is important not to confuse this with the study of abilities to learn to predict what will probably happen in a particular situation, by collecting statistics from many observations.

That ability to predict consequences of something specific is very different from the ability to detect a collection of possibilities with common abstract properties and relations, and reason about effects of realising various combinations of those possibilities, including combinations that have never previously been tried, which humans and some other species can do, but many cannot.

A tiny example: seeing an object A inside an enclosure E with one opening, and seeing that the smallest diameter of A is larger than the largest diameter of the opening supports the conclusion that no possible combination of rotations and horizontal translations of A will get it out of the enclosure E.

Current AI systems will have to sample lots of attempts to move A out of E and they can only reach a probabilistic conclusion.

They lack a generative ontology for structures and processes of varying complexity, and abilities to explore possible combinations of such structures and reason about their consequences.

The AI systems can generate and run random sample simulations, collect statistical data and reach probabilistic conclusions, but that's a totally different ability, that is a long way from the combination of information processing mechanisms required for making the discoveries of the great ancient mathematicians, including those reported in Euclid's Elements, on which I'll say more below.

The early (proto-mathematical) forms of reasoning about possible spatially constrained processes cab occur in pre-verbal human toddlers, and also in other intelligent species that do not develop anything like a human communicative language (e.g. crows, squirrels, elephants, etc.)

Evolved internal languages

So those mathematical discoveries about possible or impossible transformations do not require the use of a human communicative language.

Yet they must involve suitably structured *internal* languages used for encoding information about contents of perception, about desired states, about possible future actions, about dangers to avoid, etc.

Such languages may also support (primitive?) forms of internal mathematical reasoning about possibilities for change and possibilities for action in the environment. Their use in intelligent animals may include reasoning about partial orderings (A is getting nearer to D than B is) and about structural changes that block or open up possibilities.

(I have been collecting examples, but it is not hard to think of many more. Bearing these possibilities in mind when watching young children and other animals can be very revealing. Piaget was one of the few psychologists who understood the importance of this study of possibilities and impossibilities, compared with collecting statistical data about what happens when.)

Evolved mathematical abilities

The transition from the sorts of information processing discussed so far, to mathematical reasoning required several stages of biological evolution as well as stages of development in individuals. Some of the transitions required meta-cognitive competences: abilities to notice one's own information processing and compare different processes and their outcomes. If useful and not so useful patterns of thinking/reasoning/planning are noticed and used that can eliminate future errors.

NOTE: Kenneth Craik, in 1943 published his little book, The Nature of Explanation, in which the ability to run an internal model of aspects of the environment was emphasised. That is a sort of mathematical ability, now commonplace in many applications of computers.

He (like Gibson) appears not to have noticed the need for a more sophisticated ability to notice new possibilities for change and new combinations of possibilities for change, and to reason about their consequences, including making some things impossible. I think Immanuel Kant noticed these features of mathematical reasoning in everyday life, including reasoning about topological relationships and properties of orderings.

There are now some AI systems that can do mathematical reasoning, including reasoning about logical, numerical and algebraic relationships, in which they outstrip most humans. They usually do this using axioms and rules of inference provided by human programmers. However as far as I know they are still nowhere near replicating the ancient mathematical discovery processes, or those in very young children, squirrels, crows, etc.

Some people think its obvious that computer-based reasoners will never have such capabilities, but they usually don't have strong arguments. I have an open mind as to whether new forms of virtual machinery implemented on current computers will suffice or whether new forms of computation will be needed, that already occur in brains but have never been understood -- e.g. chemistry-based computers.

But I have jumped too fast from the earliest forms of life to human mathematical discovery and reasoning. There are very many intermediate forms all depending on requirements for control, of external actions, of information processing, of information gathering, of motive selection, of action selection, etc. And they all require mathematical reasoning abilities, even if some of them are very simple.

When processes occur control may be needed

The crucial common theme above is the need for control.

Many products of evolution have abilities to *control* structures and processes, on the basis of changing information.

This gets more complex when the same mechanisms need to have different configurations or different behaviours to deal with different environments, or to meet different needs (e.g. acquiring food, escaping predators, constructing a nest, returning to a nest, mating, etc., etc.).

Two relatively simple examples are use of information in a structure to control *formation* of a new structure, e.g. building a wall or a nest or a bone structure, and use of sensed information to control *behaviour* of an existing structure, e.g. maintaining or altering the state of a dynamical system using homeostasis (negative feedback).

All of these structures, mechanisms and processes have mathematical properties, many discovered and used blindly by evolution, and later used knowingly by some products of evolution.

So there are:

-- mathematical features of structures and processes built from physical construction kits, fundamental and derived, and

-- mathematical features of mechanisms and processes of information-based control of internal and external behaviours.

Some of these are anthropological phenomena, but by no means all. Certainly the evolved control abilities of ancient organisms are not anthropological, but they make use of mathematical structures, e.g. negative-feedback control loops, and more complex structures used in control of processes of formation of animal bodies -- still mysterious in many respects.

None of these depend on human languages, though some of them have been described and studied using human languages, including new mathematical languages.

E.g. the language of differential and integral calculus significantly extended human languages (gave them new expressive and reasoning powers), as do many mathematical, scientific, engineering and programming formalisms.

But many of the phenomena about which differentiation and integration allow us to reason existed long before Newton and Leibniz made their mathematical discoveries/inventions.

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  Also much of his earlier work, e.g. Piaget (1952) above.
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Maintained by Aaron Sloman
School of Computer Science
The University of Birmingham