



Meta-Morphogenesis and Toddler Theorems: Case Studies

Part of the [Meta-Morphogenesis](#) project
(DRAFT: Liable to change -- needs much reorganisation!)

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PARTIAL HISTORY OF THIS PAGE

Installed: 7 Oct 2011

Last updated: 25 Dec 2017 (reorganised). Format changes May 2020. 17 Nov 2020 added [Possibility comparisons](#).

19 Aug 2017 (Added new version of toddler+pencil video, with commentary).

30 Jun 2017 (Added [door-closing](#) example.)

18 May 2015 (Added [pencil/hole example](#)); 23 Jun 2015 (Intro revised); 8 Jul 2015

12 May 2015 (Considerable re-formatting)

11 May 2015

(Added linking/unlinking rings/loops example: [Impossible transitions involving rings](#))

24 Oct 2014 (Moved drawer theorem to introduction.)

18 Jun 2014 (Revised introduction. More references.); 13 Jul 2014; 25 Sep 2014; 24 Oct 2014

22 May 2014 (More references. New introduction. Some reorganisation.); 4 Jun 2014

12 Sep 2013 (reorganised, and table of contents improved); 4 Oct 2013; 19 Mar 2014

28 Sep 2012; 10 Apr 2013 (including re-formatting); 8 May 2013; 7 Aug 2013;

9 Oct 2011; 21 Oct 2011; 29 Oct 2011;; 7 Jul 2012; ... 23 Aug 2012;

This web page is

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/toddler-theorems.html>

Or: <http://goo.gl/QgZU1g>

A messy automatically generated PDF version of this file (which may be out of date) is:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/toddler-theorems.pdf>

This is one of a set of documents on meta-morphogenesis, listed in

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/meta-morphogenesis.html>

A partial index of discussion notes is in

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/AREADME.html>

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Introduction

The study of toddler theorems is the study of the variety of types of proto-mathematical learning and development in young children and other animals that in humans are the precursors of explicit mathematical competences and achievements. The transition from proto-mathematical understanding to mathematical understanding seems to require at least a meta-cognitive "layer", and later on several "stacked" meta-cognitive layers, in the developing information-processing architecture.

Such meta-cognitive layers, allowing what is and is not known to be noticed and thought about, may not be available at birth, but seem to develop later on (at various stages) in normal humans but not in other animals, though some other animals may have partial forms. But crucial proto-theorems may be discovered without such meta-meta-cognition, and used, unwittingly, and often without being noticed by doting parents and researchers.

The claim is that even pre-verbal toddlers can make discoveries about what is and is not possible in various situations, and put those discoveries to use, but without knowing they are doing that. This is a deeper and, for humans, more important ability than the ability to acquire statistics-based abilities to predict what is very likely or very unlikely. Sets of possibilities are logically, metaphysically, and cognitively prior to probabilities -- a claim that will be discussed in another document later.

A core hypothesis is that there are important forms of learning that involve being able to discover [sets of possibilities \(Piaget 1981\)](#) inherent in a situation and their [constraints or necessary connections \(Piaget 1983\)](#). This is a much deeper aspect of intelligent cognition than discovery of correlations, as in reinforcement learning, e.g. using Bayesian nets. (Here's a simple tutorial [Bayes Nets](#).)

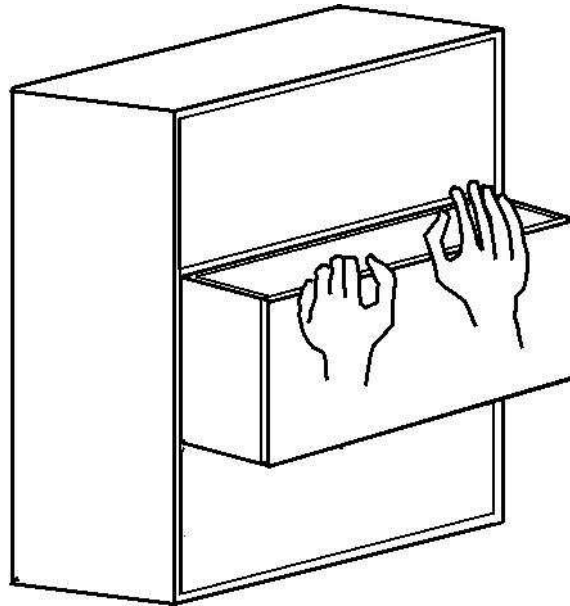
Example: The drawer-shutting theorem

Several years ago, Manfred Kerber reported that one of his children, when very young, developed a liking for shutting open drawers.

He would put both hands on the rim of the open drawer and push: OUCH!

Eventually he discovered a different way that avoided the pain.

If you push a close-fitting drawer shut with your fingers curled over the top edge your fingers will be squashed, because, although it is possible for the open-drawer to be pushed towards the shut position, it is impossible for it to avoid squashing the curled fingers (if they stay curled during pushing.)



On which hand will the fingers be squashed when the drawer is pushed shut?
(Figure added 14 Oct 2014)
(Apologies for low quality art.)

Is the discovery that using the flat of your hand to push a drawer shut avoids the pain a purely empirical discovery? Or could the consequence be something that is **worked out**, either before or after the action is first performed that way. Perhaps that is a toddler theorem -- for some toddlers?

What sorts of representational, architectural, and reasoning (information manipulation) capabilities could enable a child to work out

- WHY pushing the drawer the child's first way will produce pain?
- WHY pushing the second way will avoid the pain?

The answer seems to have two main aspects, one non-empirical, to do with consequences of surfaces moving towards each other with and without some object between them, and the other an empirical discovery about relationships between compression of, or impact on, a body part and pain or other experiences.

A sign that the child has discovered a **theorem** derived in a generative system, may be the ability to deal with other cases that have similar mathematical structures, despite physical and perceptual differences, e.g. avoiding trying to shut a door by grasping its vertical edge, without first trying it out and discovering the painful consequence.

Perceiving the commonality between what happens to the edge of a door as it is shut (a rotation about a vertical axis) and what happens to the edge of a drawer when it is shut (a translation in a horizontal plane) seems to require the ability to use an ontology that *goes beyond sensory-motor patterns in the brain, and refers to structures and processes in the environment: an [exosomatic ontology](#)*.

Once learnt, the key facts can be abstracted from drawers and horizontal edges and applied to very different situations where two surfaces move together with something in between, e.g. a vertical door edge. As Immanuel Kant pointed out in 1781, the mathematical discoveries may initially be triggered by experience, but that does not make what is learnt empirical, unlike, for example, learning that pushing a light switch down turns on a light. No matter how many examples are found without exceptions this does not reveal a necessary connection between the two events. Learning about electrical circuits can transform that knowledge, however.

There seem to be many different domains in which young children can acquire perceptual and cognitive abilities, later followed by development of meta-cognitive discoveries about what has previously been learnt, often resulting in something deeper, more general, and more powerful than the results of empirical learning. The best known example is the transition in young children from pattern-based language use to grammar based use, usually followed by a later transition to accommodate exceptions to the grammar. Like Annette Karmiloff-Smith, whose ideas about 'representational re-description' are mentioned [below](#), I think this sort of transition (not always followed by an extension to deal with counter examples) happens in connection with many different domains in which children (and other animals) gain expertise. Moreover, as proposed in a theory developed with [Jackie Chappell \(2007\)](#) and illustrated below in [Figure Evo-Devo](#), this requires powerful support from the genome, at various stages during individual development.

The mathematical and proto-mathematical learning discussed in this document cannot be explained by the statistical mechanisms for acquiring probabilistic information now widely discussed and used in AI, Robotics, psychology and neuroscience. Evolution discovered something far more powerful, which we do not yet understand. Some philosophers think all mathematical discoveries are based on use of logic, but many examples of geometrical and topological reasoning cannot be expressed in logic, and in any case were reported in Euclid's *Elements* over two thousand years ago, long before the powerful forms of modern logic had been invented by Frege and others in the 19th Century. I'll make some suggestions about mechanisms later. Building and testing suitable working models will require major advances in Artificial Intelligence with deep implications for neuroscience and philosophy of mathematics.

Re-formulating an empirical discovery into a discovery of an impossibility or a necessary connection is sometimes more difficult than the drawer case (e.g. you can't arrange 11 blocks into an $N \times M$ regular array of blocks, with N and M both greater than 1 -- why not?). Different mechanisms may have evolved at different stages, and perhaps in different species, for making proto-mathematical discoveries. Transformations of empirical discoveries into a kind of mathematical understanding probably happens far more often than anyone has noticed, and probably take more different forms than anyone has noticed. They seem to be special subsets of what Annette Karmiloff-Smith calls "Representational Redescription", also investigated by Jean Piaget in his last two books, on [Possibility](#) and [Necessity](#).

Proto-mathematical understanding may be acquired and used without the learner being aware of what's happening. Later on, various levels and types of meta-cognitive competence develop, including the ability to think about, talk about, ask questions about and in some cases also to teach others what the individual has learnt. All of this depends on forms information processing "discovered" long ago by the mechanisms of biological evolution but not yet understood by scientists and philosophers of mathematics, even though they use the mechanisms. Arguments that languages and forms of reasoning must have evolved initially for internal, "private", use rather than for communication can be found in [Talk 111](#).

The aim of this document is mainly to collect examples to be found during development of young children. Discussions of more complex examples, and requirements for explanatory mechanisms, can be found in other documents on this web site. This one of many strands in the [Meta-Morphogenesis project](#).

Added 18 Jun 2014:

One problem for this research is that it can't be done by most academic developmental psychologists because the research requires detailed, extended, observation of individuals, not in order to discover **regularities** in child cognition and development, but in order to discover what sorts of capabilities and changes in capabilities **can** occur. This is a first step to finding out what sorts of mechanisms can **explain** how those capabilities and changes are possible (using the methodology in [chapter 2](#) of Sloman (1978), expanded in [this document](#) on explaining possibilities. This requires the researchers to have kinds of model-building expertise that are not usually taught in psychology degrees. (There are some exceptions, though often the modelling tools used are not up to the task, e.g. if the tools are designed for numerical modelling and the subject matter requires symbolic modelling.)

This is not regarded as scientific research by a profession many of whose members believe (mistakenly) the Popperian myth that the only reportable scientific results in psychology must be regularities observed across members of a population, and where perfect regularities don't exist because individuals differ, then changes in *averages* and other statistics should be reported.

In part that narrow, unscientific mode of thinking is based on a partial understanding of the emphasis on falsifiability in Karl Popper's philosophy of science, which has done a lot of harm in science education. What is important in Popper's work is the idea that explanatory theories should have consequences that are as precise and general as possible. But they may not be falsifiable for a long time because the theory does not entail regularities in observables, and does not make predictions about *all* or even some *proportion* of learners.

Instead it may successfully guide searches for new, previously unnoticed, types of example covered by those possibilities. A later development of the theory could provide suggestions regarding explanatory mechanisms. For such mechanisms it is more important to produce working models demonstrating the potential of the theory than to use the theory to make predictions. Such research sometimes gains more from detailed long term study of individuals, and speculative model building and testing, than from collection of shallow data from large samples.

For more on the scientific importance of theories explaining how something is possible see <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/explaining-possibility.html>
Construction kits as explanations of possibilities

Teaching based on a deep theory may make a huge difference to the performance of a small subset of high ability learners even if the theory does not specify how those learners can be identified in advance as a basis for making predictions. Moreover the theory may explain the possibility of a variety of developmental trajectories that can be observed by good researchers when they occur, though theory may not (yet) give clues as to which individuals will follow which trajectories. Many biological theories have that form, e.g. explaining how some developmental abnormalities can arise without being rich enough to predict which individual cases will arise. In some cases that may be impossible in principle if the abnormalities depend on random chemical or metabolic co-occurrences during development about which little is known.

A theory explaining how sophisticated mathematical competences **can** develop may make no falsifiable predictions because there are no regularities -- especially with current teaching of mathematics in most schools. (Unless I've been misinformed.)

[27 Sep 2014: To be expanded, including illustrations from linguistic theory.]

One of the bad effects of these fashions is that the only kind of recommendation for educational strategies such a researcher can make to governments and teachers is a recommendation based on evidence about what works for all learners, or, failing that, what works for a substantial majority of learners.

(E.g. a recommendation to teach reading using *only* the phonic method -- which assumes that the main function of reading is to generate a mapping from text to sounds, building on the prior mapping from sounds to meanings. That recommendation ignores the long term importance of building up direct mappings from text to meanings operating in parallel with the mapping from sounds to meanings, and construction of architectural components not required for reading out loud but important for other activities later on, e.g. inventing stories or hypothetical explanations.)

Another bad effect of the emphasis on discovering and reporting what normally **does** happen rather than what **can** happen is to deprive psychology of explanatory theories able to deal with outliers, such as Bach, Mozart, Galileo, Shakespeare, Leibniz, Newton, Einstein, Ramanujan, and others. In contrast, a deep theory about what is possible and how it is possible can account both for what is common and what is uncommon, just as a theory about the grammatical structure of English can explain both common utterances and sentences that are uttered only once, like this one.

A tentative proposal:

The examples of toddler theorem discoveries given below are isolated reports of phenomena noticed by me and various colleagues, along with cases presented in text books, news reports or amateur videos on social media. Perhaps this web site should be augmented with a web site where anyone can post examples, and where development of individual babies, toddlers and children over minutes, hours, days, weeks, months or years can be reported. Something like [Citizen Science](#) for developmental psychology? Any offers to set that up?

More examples are presented below.

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1. THIS DOCUMENT

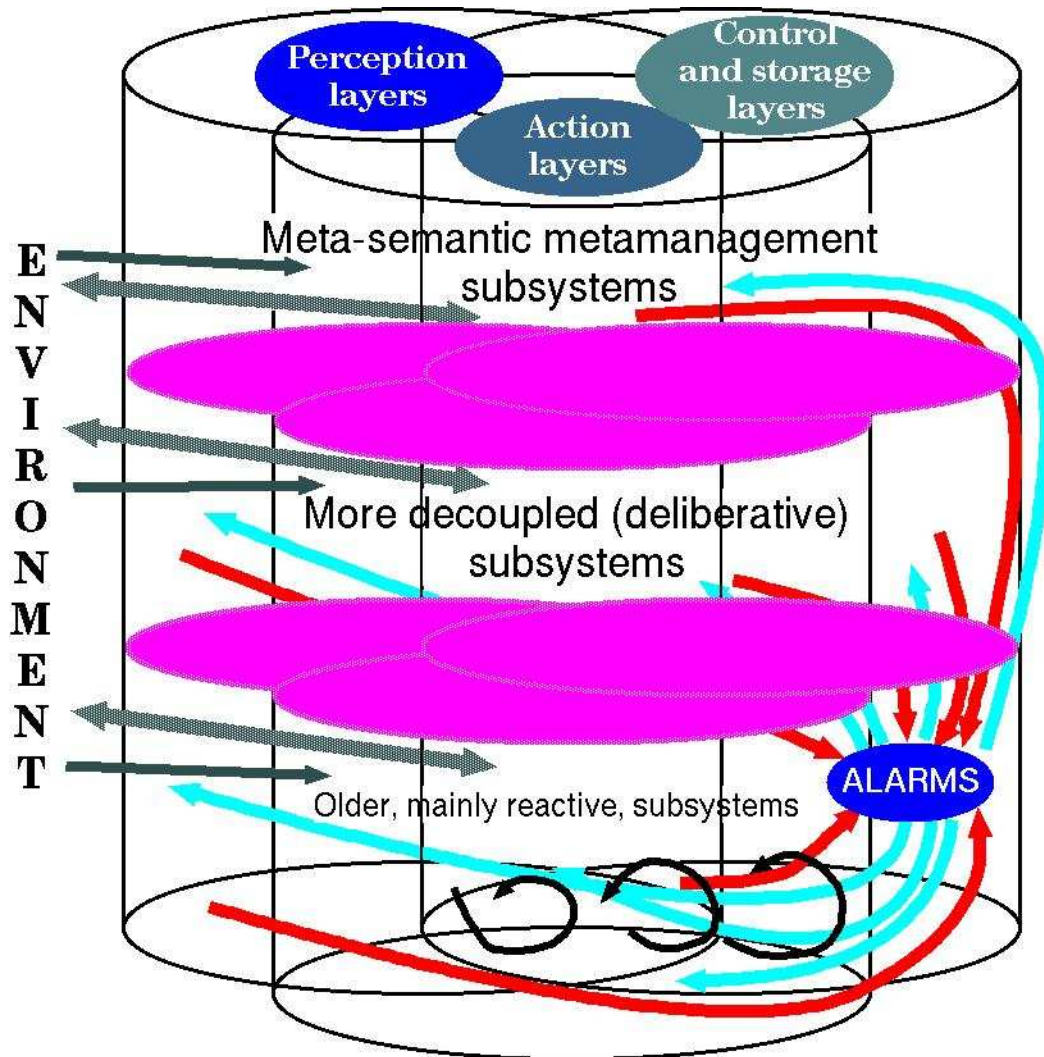
This document reports cases of observed or conjectured discoveries of toddler theorems by children of various ages. Ideally such a survey should be developed in the context of a theoretical background that might include the following items:

- A theory of the types of information processing architectures that can exist at different stages of development of intelligent individuals. This might include an abstract architecture schema covering a wide range of possible information-processing architectures and a wide range of possible requirements for developing intelligent animals or machines so that different architectures and different sets of requirements (niches) can be located in that framework. A possible framework, still requiring much work, is summarised in

<http://www.cs.bham.ac.uk/research/projects/cogaff/#overview>.

Some of the components and functions required in animal or robot information processing architectures are crudely depicted and sub-divided in the figure below, where processes and mechanisms at lower levels are generally evolutionarily much older than those at higher levels, and probably develop earlier in each individual, though new ones may be added later through training:

Figure CogArch



(Recently revised diagram of CogAff Schema, thanks to Dean Petters.)

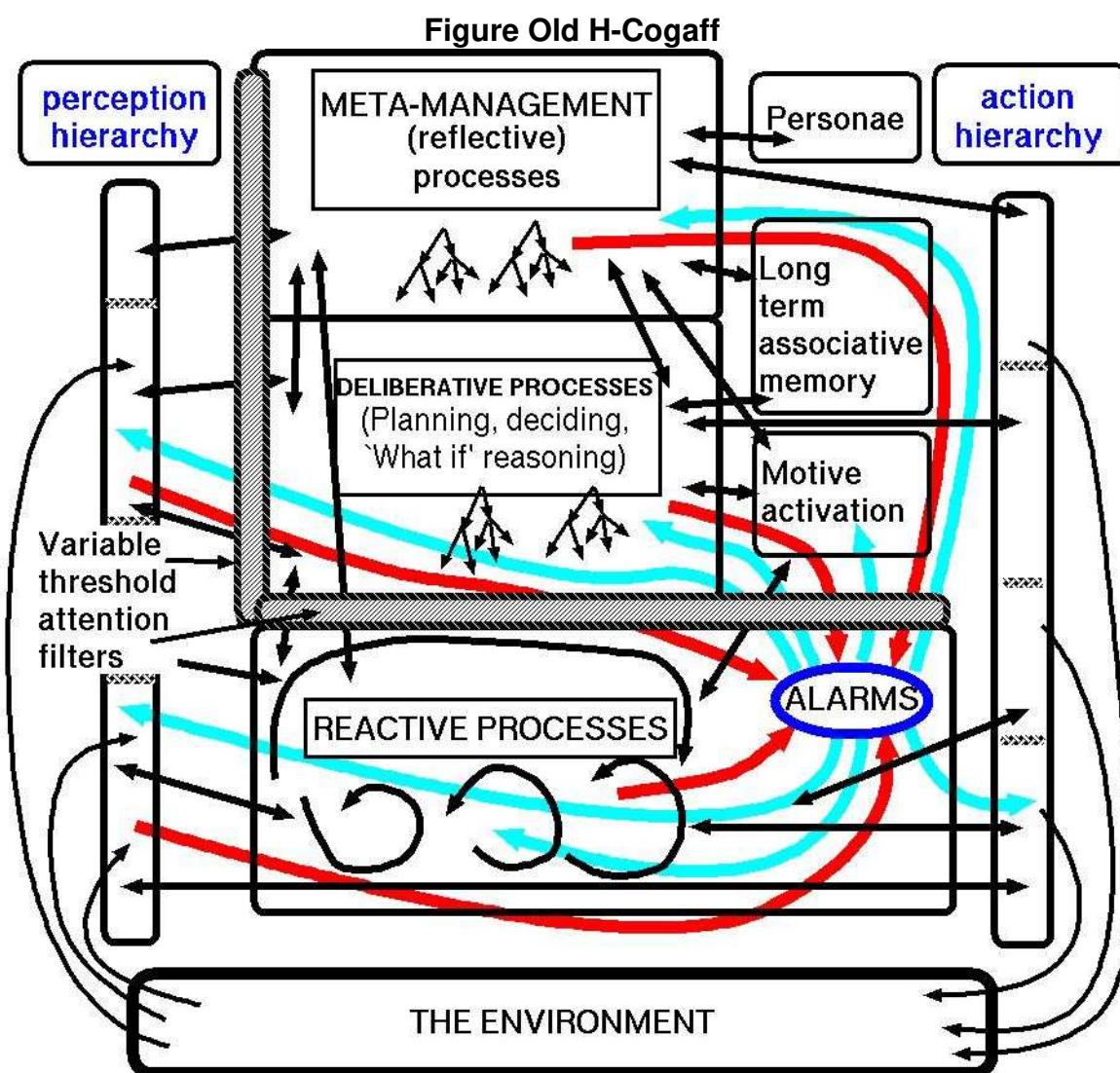
Note: the above diagram simplifies many important features of required architectures, including the "alarm" processing routes and mechanisms described in other CogAff papers (allowing asynchronous interruption or modulation of ongoing processes, e.g. to meet sudden threats, opportunities, etc.) Mechanisms related to use of language are distributed over all the functional subdivisions between columns and layers.

The architectural ideas are discussed in relation to requirements for virtual machinery here: <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/vm-functionalism.html> Including an older version of the Human Cogaff (h-cogaff) diagram, namely

An older version: the H-Cogaff architecture

NOTE: added here 6 Feb 2021

This version includes "personae": a collection of personalities that can be available to take control of the system, according to context, with various relationships between them, including competition in some pathologies. It also did not bring out the overlaps between perception and action indicated in the previous diagram (e.g. sensing a surface texture can involve sliding a finger along it, sensing weight can involve lifting or pushing the object sensed).



The development of proto-mathematical and mathematical competences listed below make use of mechanisms, including changing mechanisms, in all the layers and columns of mechanisms depicted in the above diagrams. No diagram, however, can adequately represent the richness and diversity of components and the functionality they add. no

Note: 25 Dec 2017

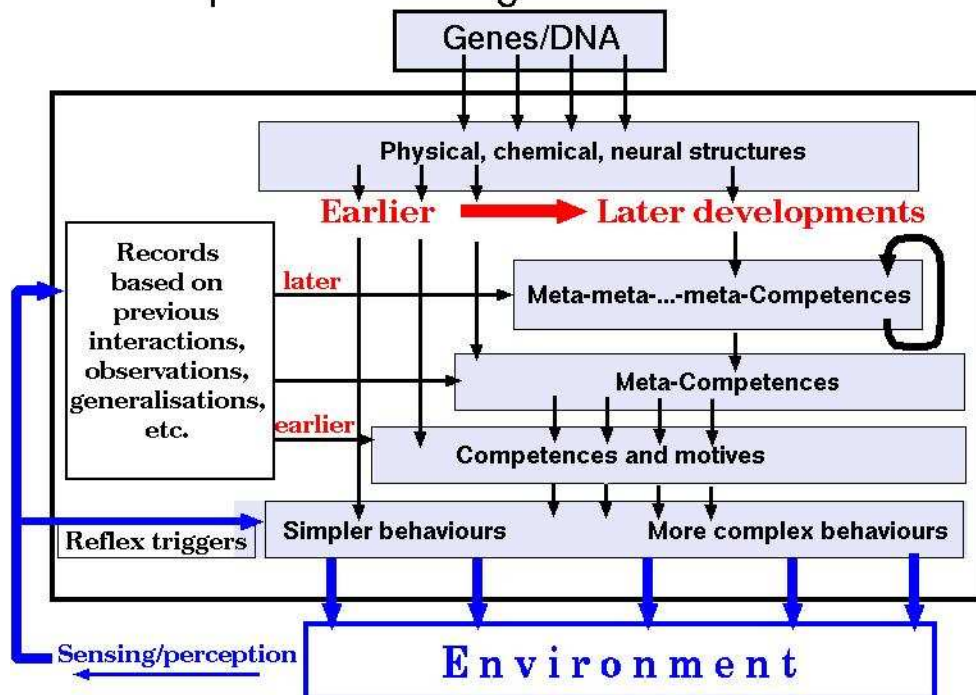
After collecting many examples of competences to be explained, especially the competences involved in ancient discoveries in geometry and topology, long before the development of the modern logic-based axiomatic method and use of Cartesian coordinates to represent geometry, I have begun to explore the possibility that a kind of "Super-Turing"

information processing mechanism must have been produced by evolution. The ideas will be elaborated in

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/super-turing-geom.html>

- A theory of types of information processing mechanism available at various stages during the individual's development at various stages of evolution. Clearly the *initial* mechanisms (in a fertilised egg or seed) are purely chemical. In many organisms, though not the majority, nervous systems of various sorts are grown under the control of complex information processing mechanisms that are slowly being unravelled. The nervous systems provide new mechanisms that continue to develop themselves and control more and more biological functions. Although there have been tremendous advances in our knowledge I think there may still be far more to be discovered in the remainder of this century than has already been learned. In particular, as hinted by Turing in his 1950 paper, brains may make far more important use of chemical information processing than has so far been noticed.
- A schematic theory of iteratively developed, increasingly sophisticated, types of interactions between genome and environment during individual development -- forming increasingly complex domains of competence. A first draft theory of this type is outlined in [Chappell & Sloman \(2007\)](#), which included an earlier version of this diagram crudely summarising interactions between genome and environment:

FIG EVO-DEVO: The Meta-Configured Genome (MCG)
Multiple routes from genome to behaviours



[New version of diagram installed here: 12 May 2015]

[Chris Miall helped with the original version of this diagram.]

Compare Waddington's "Epigenetic Landscape". Our proposal is that for some altricial species developmental processes rebuild or extend the landscape at various stages during development, and then choose newly available developmental routes after rebuilding, instead of merely choosing a trajectory on a fixed epigenetic landscape.

For later work on the MCG theory (including video) follow this link
<https://www.cs.bham.ac.uk/research/projects/cogaff/movies/meta-config/>

One of the features of a system like this is that if the stages are extended in time, and if the earlier stages include development of abilities to communicate with conspecifics and acquire information from them, then later developments (to the right of the diagram) can be influenced by not only the physical and biological environment as in other altricial species, but also by a culture.

As we see on this planet, that can have good effects, such as allowing cultures to acquire more and more knowledge and skill, and bad effects such as allowing religious ideas, cruel practices, superstition, and in some cases "mind-binding" processes that prevent the full use of human developmental potential, as discussed in:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/teaching-intelligent-design.html#softwarebug>
'Religion as a software bug'

Note:

I hope to show later on how the above model of interactions between genome and environment in individual members of advanced species can be modified to produce a partly analogous model of how evolution works within a portion of the physical universe. Both are examples of dynamical systems with creative powers, able to transform themselves not merely by adjusting numerical parameters but by introducing new abstract types of structure and types of causal power, which can later be instantiated in different ways in different contexts. This can be seen as partly analogous to abductive reasoning, in which evidence inspires formulation of a new explanatory hypothesis that is added to previous theories, in some cases with new undefined symbols that "grow" semantic content through deployment of the theory. See <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/meta-configured-genome.html>

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2. WHAT ARE TODDLER THEOREMS?

Here are some facts whose significance does not seem to be widely appreciated:

- Many non-human animal species have cognitive abilities (including perceptual abilities) that require the use of a rich expressive internal language with generative power and compositional semantics.
- The same is true of pre-verbal children, though not all of the mechanism exists at birth: there is a process of growth of the information-processing architecture driven partly by the environment and partly by the genome.
- In both cases there is a type of learning that is not included in the standard taxonomies of learning (from observation, from statistical relationships, from experiment, from imitation, from instruction by others), namely a process of learning by **working things out** which in adult humans most obviously characterises mathematical discoveries, including discovery or creation of
 - new powerful concepts (extending the ontology used)
 - new powerful notations (formalisms)
 - new forms of calculation or reasoning
 - new conjectures
 - new proofs

- new implications of what was previously known
- The biological precursors of abilities to do mathematics explicitly are mechanisms that allow animals and very young children to solve practical problems, including novel problems, without going through lengthy processes of trial and error and without having to take risky actions whose possible consequences are unknown -- a capability famously conjectured by Kenneth Craik in *The Nature of Explanation* (1943).
- Although Immanuel Kant, Max Wertheimer, Jean Piaget, Konrad Lorenz, Lev Vygotsky, John Holt, and others noticed examples of this kind of ability in human learners, and sometimes in other species, the links with adult mathematical competences are unclear, and as far as I know have not been studied or modelled.
- I suggest that without these ancient biological capabilities humans could never have made the discoveries that were later organised cooperatively in systems of knowledge, such as Euclid's Elements (whose contents and methods are unfortunately no longer a standard part of the education of bright children -- with dire consequences for many academic disciplines, including psychology and education).
- I suspect that many of the more basic ancient discoveries, and others that have never been documented, are repeated by young children without anyone noticing. A few years ago I started using the label "toddler theorem" to express this idea, though I don't think the discoveries are restricted to the age-range normally covered by the label "toddler". However, the mechanisms required are probably not all available at birth: evolution discovered the benefits of delaying development of meta-cognition until a substantive collection of information had been acquired (explained in more detail in [Chappell and Sloman \(2007\)](#)).
- In fact, the discovery processes can continue throughout life and lead to many solutions to practical problems as well as advances in engineering, science and mathematics, though individuals vary in what they can achieve, and the extent to which they use the potential they have (education can be very damaging in this respect).
- Many of the discovery processes appear to be examples of what [Annette Karmiloff-Smith](#) has called "Representational Redescription", summarised in this (incomplete) introduction to her work:
<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/beyond-modularity.html>
- There are deep implications for philosophy of mathematics, including the problems I addressed in my DPhil thesis ([1962](#)), which was an attempt to defend Kant's philosophy of mathematics.
- I think some aspects of the forms of reasoning used in the discovery of toddler theorems are not yet represented in AI or robotic systems, and it may even be very difficult or impossible to implement some of them on Turing machines and digital computers because they use the interplay between continuous and discrete structures and processes. Readers who have no idea what I am talking about may find it helpful to look at some examples, e.g. a discussion of some of what can be learnt by [playing with triangles](#).
- The main aim of this web site is to introduce the idea of a "toddler theorem" and present examples. I suspect that with help from observant parents, grandparents, teachers, and animal cognition researchers, the list of examples of "toddler theorems" should grow to include many hundreds of types of example.
- Since many theorems involve a domain (a class of structures or processes or

relationships) I have also included [below](#) a brief discussion of the concept of domain (also used by Karmiloff-Smith and many others, though with varying terminology). The ideas are developed at greater length in this discussion document

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/bio-math-phil.html> and in this presentation to the PT-AI 2013 conference: <http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#talk108>

Also closely related, is [this presentation](#) on the roles of richly structured internal languages, and why they must have evolved before languages for communication, and why they need to develop in advance of use of language for communication.

3. NOTE: The word 'toddler' can be interpreted broadly for our purposes:

3.a. The Toe-ball example

Pre-toddler theorems? (Added 24 Sep 2014)



For example, this 11-month old child is not a toddler, as she cannot yet walk, and has recently learnt to crawl, but she seems to have made a discovery about things that can be supported between upward pointing toes and downward facing palm.

Whether that's a "theorem" for her depends on whether she was able (using whatever representational resources are available to a pre-verbal human) to reason about the consequences of previously acquired information about affordances so as to predict what would happen in this novel situation, or retrospectively to understand why it happens if it first happened unintentionally.

Clearly whatever initiated the processes she continued it intentionally and even seemed to be trying to share what she had discovered with someone not in the picture. The differences between possible cases need further investigation elsewhere. There are also many examples involving actions that produce changes of posture (e.g. from lying on back or belly to sitting upright) and various crawling actions that provide forward or backward motion or change of direction.

As for why children do such things, I believe the normal assumption that all motivation must be reward based is false, as discussed below in the section on [Architecture-Based motivation](#).

Another pre-toddler-theorem in the case of this child seems to be that the transition between -- crawling forward with legs stretched backward (position a, below), and -- sitting on the ground with legs projecting forward and facing roughly in the original position (position d) can be achieved by temporarily extending legs sideways, aligned as in a hinge joint, as illustrated by positions (b) and (c) in the sequence below. She also uses the same intermediate state for the reverse transition. (The much more common strategy involves rolling

over on one side before or after changing direction.)



(a)

(b)



(c)

(d)

I would be grateful for information about any other infants who use this or a related method for doing the 90 degree rotation of torso and roughly 180 degree rotation of legs.

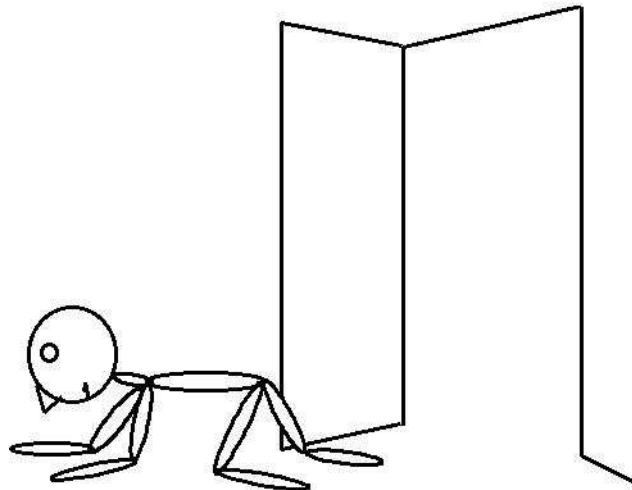
3.b. A crawler's door-closing theorem

(Added 1 Jul 2017) This is based on a recollected episode over a decade ago, when a baby and his parents were visiting us. At one point he crawled from the front hall into an adjoining room, indicating that he wanted me to follow him (e.g. stopping, waiting and looking round at me if I paused while following him). After he had crawled through the door and waited for me to follow him, he wanted the door shut. (I have no idea why, perhaps he had no reason.) He managed to push it shut with his feet, after crawling to an appropriate location, rolling over onto his back, swinging his legs back round the door, then pushing shut.

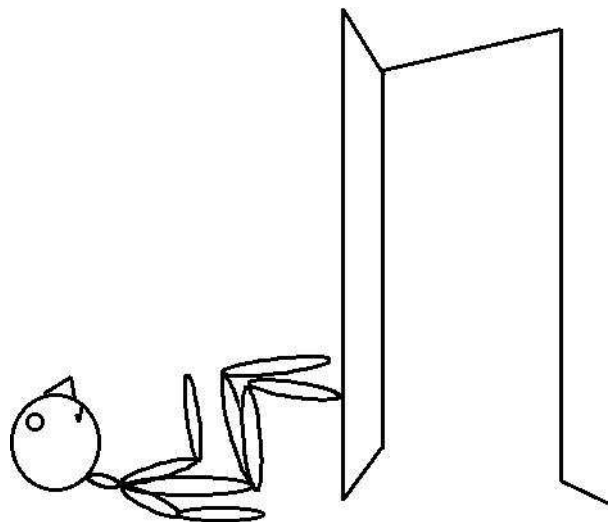
That action can be thought of as a proof (by construction) of the theorem that it is possible to shut a door with your feet after crawling through the doorway.

How was the intention to do all that represented in his brain (or mind) long before he could say anything in words?

Fig: Crawler
Crawler works out how to shut door after crawling through it



Crawls through open door facing into room



Rolls over onto back to push door shut with feet.

At the time, I did not think of asking his parents whether he had been taught to do that, or had regularly been doing it at home. In either case he seemed to understand what he was doing, and was able to manoeuvre into the right position, to get the door shut the first time he tried in our house, which had a very different layout from his home.

What kind of representation of spatial structures, relationships, and possibilities for change could a child's brain use (a) in forming the intention to perform such an action, and (b) in actually doing it? I suspect the answer will refer to precursors to the mechanisms that enabled ancient mathematicians to make profound mathematical discoveries.

See also: <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/impossible.html>

4. BACKGROUND

- **Philosophy of Mathematics, AI, Representational Redescription and Toddler Theorems**

There are problems about human spatial reasoning abilities and other non-logical reasoning abilities that I started thinking about when working on my DPhil in Philosophy of Mathematics, Oxford 1962

```
"Knowing and Understanding:  
  Relations between  
    meaning and truth,  
    meaning and necessary truth,  
    meaning and synthetic necessary truth
```

<http://www.cs.bham.ac.uk/research/projects/cogaff/sloman-1962>

(Digitised version installed 2016).

This argued (e.g. against Hume) that Immanuel Kant was right in claiming in 1781 that in addition to

1. true empirical propositions that in principle could be refuted in experiments and observations with novel conditions and
2. analytic, essentially trivial, truths that depend only on definitions and their logical consequences, and whose discovery does not extend factual knowledge, apart from knowledge of logical consequences of collections of definitions, including unobvious consequences,

there are also truths that are neither empirical nor trivial but provide substantial knowledge, namely synthetic, necessary, truths of mathematics, whose discovery requires non-empirical reasoning capabilities.

Some of the concepts used here are explained in this summary of parts of my DPhil thesis:

"'NECESSARY', 'A PRIORI' AND 'ANALYTIC'" (1965)

<http://www.cs.bham.ac.uk/research/projects/cogaff/62-80.html#1965-02>

Two more papers based on the thesis work were published in 1965 and 1969:

<http://www.cs.bham.ac.uk/research/projects/cogaff/62-80.html#rog>

Functions and Rogators (1965)

<http://www.cs.bham.ac.uk/research/projects/cogaff/62-80.html#1968-01>

Explaining Logical Necessity (1968-9)

Around 1970 Max Clowes introduced me to Artificial Intelligence, especially AI work on Machine vision. That convinced me that a good way to make progress on my problems might be to build a baby robot that could, after some initial learning about the world and what can happen in it, notice the sorts of possibilities and necessities (constraints on possibilities) that characterise mathematical discoveries. My first ever AI conference paper distinguishing "Fregean" from "Analogical" forms of representation was a start on that project, followed up in my 1978 book, especially Chapters 7 and 8.

- **Interactions between philosophy and AI: The role of intuition and non-logical reasoning in intelligence,**
Proc 2nd IJCAI, 1971, London, pp. 209--226,
<http://www.cs.bham.ac.uk/research/cogaff/04.html#200407>
- Aaron Sloman, CRP: **The Computer Revolution in Philosophy: Philosophy, Science and Models of Mind**, Harvester Press (and Humanities Press), 1978,
<http://www.cs.bham.ac.uk/research/cogaff/62-80.html#crp>

From about 1973, I was increasingly involved in AI teaching and research and also had research council funding for a project on machine vision, some results of which are summarised in chapter 9 of CRP. Later work (teaching and research) led me in several directions linking AI, Philosophy, language, forms of representation, architectures, relations between affect and cognition, vision, and robotics. Progress on the project of implementing a baby mathematician was very slow, mainly because the various problems (especially about forms of representation) turned out to be much harder than I had anticipated. Moreover, I did not find anyone else interested in the project.

In 2008 Mary Leng jolted me back into thinking about mathematics by inviting me to give a talk in a series on mathematics at Liverpool University. In that talk and in a collection of subsequent papers and presentations I tried to collect examples and arguments about how various aspects of mathematical competence could be seen to arise out of requirements for interacting with a complex, structured, changeable environment. I did not find anyone else who shared this interest, perhaps because the people I met had not spent five years between the ages of five and ten playing with meccano?

<http://www.cs.bham.ac.uk/research/projects/cosy/photos/crane/>

● **WHAT IS A DOMAIN?**

The meta-domain of meta-domains ... of domains

Note added 4 Mar 2015

I've recently added a discussion of "construction kits" produced by and used by evolution and development, including concrete construction kits, abstract construction kits and mixed construction kits. Some sorts of domain will be related to (or generated by) a particular sort of construction kit (which itself may be a mixture of simpler construction kits). For more on construction kits see:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/construction-kits.html>

(Domains are sometimes called "micro-worlds")

Added 23 Aug 2012:

Although I started this web page in October 2011, I have been working on many of these themes for many years using different terminology. E.g. some of the ideas about numbers go back to chapter 8 of my 1978 book, but that builds on my 1962 Oxford DPhil Thesis (attempting to defend Kant's philosophy of mathematics -- before I knew anything about computers or AI).

After discovering the deep overlap with ideas Annette Karmiloff-Smith (AK-S) had developed, especially in her 1992 book, which I have begun to discuss in <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/beyond-modularity.html> I thought it might be helpful to use her label "domain", instead of the collection of labels I have been playing with over several decades (some of which have been widely used in AI, others in mathematics, software engineering, etc. -- the ideas are deep and pervasive).

I can't now remember all the labels I have used, but the following can be found in some of my papers, talks, and web pages, with and without the hyphens:

```
'micro-world'  
'mini-world'  
'micro-domain'  
'micro-theory'  
'theory'  
'framework'  
'framework-theory'
```

What is a domain?

I don't think there is any clear and simple answer to that question. But this document presents several examples that differ widely in character, making it clear that domains come in different shapes and sizes, with different levels of abstraction, different kinds of complexity, different uses -- both in controlling visible behaviour and in various internal cognitive functions --, different challenges for a learner, different ways of being combined with other domains to form new domains, and conversely, different ways of being divided into sub-domains, etc.

We might try to compare different sub-fields of academic knowledge to come up with an analysis of the concept of domain, but there are many overlaps and many differences between such domains as philosophy, logic, mathematics, physics, chemistry, biology, biochemistry, zoology, botany, psychology, developmental psychology, gerontology, linguistics, history, social geography, political geography, geography, meteorology, astronomy, astrophysics,

Moreover within dynamic disciplines new domains or sub-domains often grow, or are discovered or created, some of them found to have pre-existed waiting to be noticed by researchers (e.g. planetary motions, Newtonian mechanics, chemistry, topology, the theory of recursive functions) while others are creations of individual thinkers or groups of thinkers, for example, art forms, professions (carpentry, weaving, knitting, dentistry, physiotherapy, psychotherapy, architecture, various kinds of business management, divorce law in a particular country, jewish theology, and many more). However, that distinction, between pre-existing and human-created domains, is controversial with fuzzy boundaries.

Philosophers' concepts of "natural kinds" are attempts to make some sort of sense of this, in my view largely unsatisfactory, in part because many of the examples are products of biological evolution, and some are products of those products. I suspect the idea of "naturalness" in this context is a red-herring, since the distinction between what is created and what was waiting to be discovered is unclear and there are hybrids.

- <http://plato.stanford.edu/entries/natural-kinds/>
- http://en.wikipedia.org/wiki/Natural_kind

The distinction between "logical geography" (Gilbert Ryle) and "logical topography" (me), is also relevant, explained in <http://tinyurl.com/BhamCog/misc/logical-geography.html>,

A particularly rich field of human endeavour in which hierarchies of domains are important is software engineering, and the discovery of this fact has led to the creation of various kinds of programming languages for specifying either individual domains or families of domains. For example, so-called "Object Oriented Programming" introduced notions of

classes, sub-classes, instances, and associated methods (class-specific algorithms) and inheritance mechanisms. More sophisticated OOP languages allowed multiple inheritance and generic functions (methods that are applicable to collections of things of different types and behaviour in ways that depend on what those types are).

<http://tinyurl.com/PopLog/teach/oop>

Note added 4 Mar 2015

Using the notion of construction kit presented in

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/construction-kits.html>

we can say that many domains are "generated" or "defined" by a particular type of construction kit (which may be composed of simpler construction kits). We need a more thorough survey and analysis of cases.

More generally we can say that a domain involves relationships that can hold between types of thing, and instances of those types can have various properties and can be combined in various ways to produce new things whose properties, relationships, competences and behaviours, depend on what they are composed of and how they are combined, and sometimes the context. Often mathematicians specify such domain-types without knowing (or caring) whether instances of those types actually existed in advance (e.g. David Hilbert's infinite dimensional vector spaces?) Additional domains are summarised [below](#).

Formation of a new instance of a type in a domain can include **assembling** pre-existing instances to create larger items (e.g. joining words, sentences, lego bricks, meccano parts, dance steps, building materials, mathematical derivations), or can include **inserting** new entities within an existing structure, or changing properties, or altering relationships. E.g. loosening a screw in a meccano crane can sometimes introduce a new rotational degree of freedom for a part.

Some domains allow continuous change, e.g. growth, linear motion, rotation, bending, twisting, moving closer, altering an angle, increasing or decreasing overlap, changing alignment, getting louder, changing timbre, changing colour, and many more (e.g. try watching clouds, fast running rivers, kittens playing, ...). Some allow only discrete changes, e.g. construction of logical or algebraic formulae, or formal derivations, operations in a digital computer, operations in most computational virtual machines (e.g. a Java or lisp virtual machine), some social relations (e.g. being married to, being a client of, etc).

The world of a human child presents a huge variety of very different sorts of domains to be explored, created, modified, disassembled, recombined, and used in many practical applications. This is also true of many other animals. Some species come with a fixed, genetically determined, collection of domain related competences, while others have fixed frameworks that can be instantiated differently by individuals, according to what sorts of instances are in the environment, whereas humans and others (often called "altricial" species) have mechanisms for extending their frameworks as a result of what they encounter in their individual lives -- examples being learning and inventing languages, games, art forms, branches of mathematics, types of shelter, and many more. This diversity of content, and the diversity of mixtures of interacting genetic, developmental and learning mechanisms was discussed in more detail in two papers written with Jackie

Chappell, one published in 2005 and an elaborated version [in 2007](#). There are complicated relationships with the ideas of AK-S, which still need to be sorted out.

Tarskian model theory <http://plato.stanford.edu/entries/model-theory/> is also relevant. Several computer scientists have developed theories about theories that should be relevant to clarifying some of these issues, e.g. Goguen, Burstall and others (for example, see [http://en.wikipedia.org/wiki/Institution_\(computer_science\)](http://en.wikipedia.org/wiki/Institution_(computer_science))).

At some future time I need to investigate the relationships. However, I don't know whether they include domains that allow (continuous representations of) continuous changes, essential in Euclidean geometry, Newtonian mechanics, and some aspects of biology.

I don't know if anyone has good theories about discovery, creation, combination, and uses of domains in more or less intelligent agents, including a distinction between having behavioural competence within a domain, having a generative grasp of the domain, and having meta-cognitive knowledge about that competence. These distinctions are important in the work of AK-S, though she doesn't always use the same terminology.

The rest of this discussion note presents a scruffy collection of examples of domains relevant to what human toddlers (and some other animals and older humans) are capable of learning and doing in various sorts of domains whose instances they interact with, either physically or intellectually. The section on [Learning about numbers \(Numerosity, cardinality, order, etc.\)](#) includes examples of interconnected domains, though not all the relationships are spelled out here.

Theorems about domains are of many kinds. Often they are about invariants of a set of possible configurations or processes within a domain (e.g. "the motion at the far end of a lever is always smaller than the motion at the near end if the pivot is nearer the far end", "moving towards an open doorway increases what is visible through the doorway, and moving away decreases what is visible"). (See the section on [epistemic affordances](#), below.)

We need a more developed theory about the types of theorems available to toddlers and others to discover, when exploring various kinds of environment, and about the information-processing mechanisms that produce what AK-S calls "representational redescription" allowing the theorems to be discovered and deployed. (I think architectural changes are needed in many cases.)

[CONTENTS](#)

5. **BASICS OF THE THEORY**

Core ideas (no claims are made here about novelty):

- **Transitions in information-processing**

There are many transitions in living systems, both continuous and discrete, on various scales: within organisms, within a species, within ecosystems, within societies, or sub-cultures, etc. The obvious transitions include physical morphology and observable behaviours.

There are also transitions in information-processing capabilities and mechanisms that are much harder to detect, though their consequences may include observable behaviours.

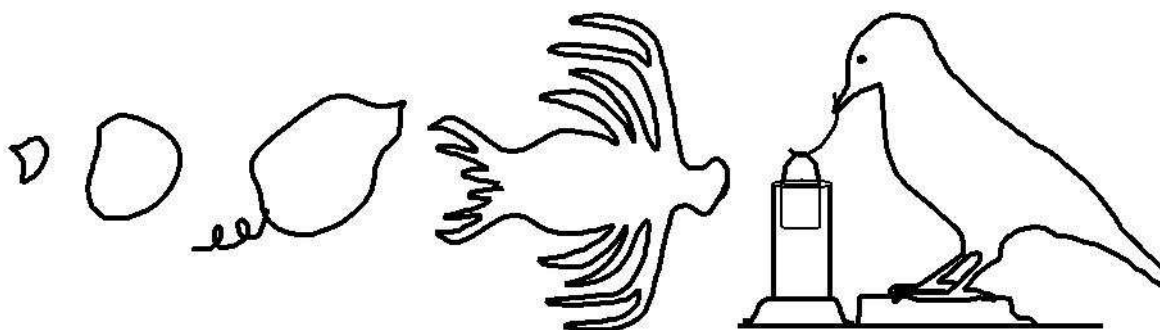
A draft (incomplete, messy and growing) list of transitions in biological information processing is [here](#).

The transitions producing new capabilities and mechanisms are examples of a generalised concept of morphogenesis, originally restricted to transitions producing physical structures and properties.

Among the transitions are changes in the mechanisms for producing morphogenesis. These are examples of meta-morphogenesis (MM). The examples of information processing competence described here may occur at various stages during the lives of individuals. The mechanisms that produce new ways of acquiring or extending competences are mechanisms of meta-morphogenesis, about which little is known. Piaget identified many of the transitions in children he observed, and thought that qualitative changes in competence producing competences were global, occurring in succession, at different ages, during the development of a child. Karmiloff-Smith, in *Beyond Modularity* suggests that transitions between stages may occur within different domains of competence, and will often be more a function of the nature of the domain than the age of the child, though she allowed that there are also some age-related changes. See <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/beyond-modularity.html>

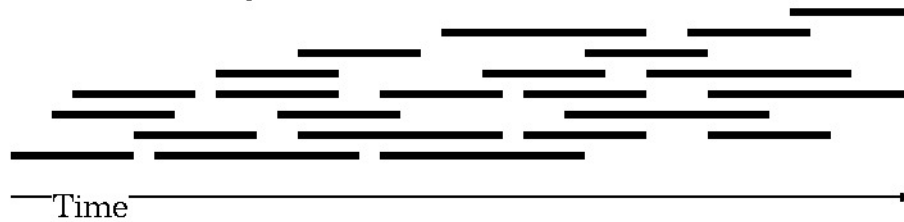
I have no idea what Karmiloff-Smith would think of my proposal to extend this idea to regarding biological evolution (i.e. natural selection) as (unwittingly) *making discoveries* about domains of mathematical structures then *transforming those discoveries* in various ways, as outlined in a separate document on the nature of mathematics and the relevance of mathematical domains to evolution and in a presentation to the PT-AI 2013 conference: <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/bio-math-phil.html>
<http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#talk108>

Mary Leng has made related claims related to my topic, but disagreeing with my claims, as reported in this book review: <http://www.ams.org/notices/201305/rnoti-p592.pdf>



Transitions occur across species, within a species, within an individual, concurrently in different species, and in some cases in eco-systems or sub-systems involving more than one species.

Evolutionary transitions



Transitions in development and learning

Transitions in domains of competence

Micro-transitions in a domain

Transformation of a domain

A draft (growing) list of significant transitions in types of information-processing in organisms is here:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/evolution-info-transitions.html>

- **It can be very hard to detect or characterise changes in INFORMATION PROCESSING CAPABILITIES, e.g. functions, mechanisms, forms of representation, architectures, ontologies used,**

People who have not designed, tested or debugged working systems may lack the concepts and theories required.

- **Turing's idea (large structures from small) -- illustrated in several areas:**
 - If the right kinds of small pieces are put together in the right kinds of ways --- then qualitatively new structures and behaviours can emerge from their interactions.
--- e.g. micro manipulations add up to proofs of mathematical theorems
 - The meta-morphogenesis project attempts to apply that idea to varieties of information processing.
 - Turing's most famous work focused on intrinsic information processing: E.g. operations in a Turing machine not connected to anything else
 - To study biological information processing we need to think about connections with an **environment**
- **Exploration-based learning**
Children and other animals do a lot of empirical exploration of their environment. The kind of exploration depends on the species, is very much influenced by what's in the environment (e.g. including clothing and toys), and also changes with age and cognitive

sophistication. It may also be partly influenced by the individual's genetic endowment.

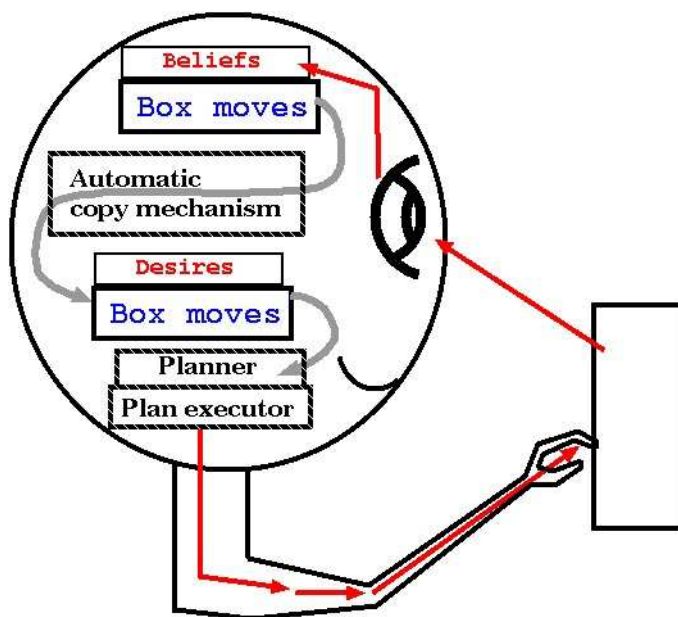
Sauvy and Sauvy(1974).]

- **Architecture-Based motivation**

Many researchers, including many (or most?) robotics researchers, believe that it is impossible to have a motive, to want to do or achieve, or prevent, or preserve something in the environment, or in thought, unless achieving that motive produces another effect which is providing a reward, which is usually a scalar quantity so that it can vary in one dimension, with the effect of increasing or reducing the probability that some preceding action will be repeated in similar circumstances. It is normally assumed that without some expected reward an animal or intelligent machine cannot possibly want to do something. (This is also an old debate in philosophy, e.g. see G.E.M.Anscome **Intention** 1957.)

I (and probably others using different terminology) have proposed that although rewards of many kinds (including non-scalar rewards) can be important, there are also non-reward-based forms of motivation, without which a great deal of the learning done by young children (and other animals) would be impossible. That's because the learner is required to select things to do without being in a position to have any knowledge about the possible outcomes. So natural selection has somehow provided motivation triggers that are directly activated by perceived states of affairs or processes, or in some cases thoughts, to create motives, which then may or may not produce behaviours, depending on which other motives are currently active, and other factors. Such a mechanism can produce forms of exploration-based learning that would otherwise not occur. I call that "architecture-based motivation" in contrast with reward-based motivation, as explained in <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/architecture-based-motivation.html>

The diagram illustrates, schematically, a very simple architecture with motives triggered by what is perceived, but with no computation of, or comparison of, rewards, or expected utility.



In particular, the individual may be unaware of what is being done or why it is being done.

I am not saying that that's a model of human or animal motive-generation, but that something with those features could usefully be an important **part** of a motive generation mechanisms if the genetically determined motive generating reflexes are selected (by evolution) for their later usefulness in ways that the individual cannot understand. This idea was independently developed and tested in a working computer model, reported by [Emre Ugur \(2010\)](#).

- **More on domains (introduced above) in learning**

In doing that exploration, individuals somehow divide up the world into (nested and overlapping) "domains" or "micro-domains", each containing some collection of (relatively) simple entities, properties, relationships, and more complex structures formed from such entities, and also simple processes in which objects change properties and relationships, along with more complex processes created by combining simpler processes, so that new structures are built, old structures disassembled, or multiple relationships changed in parallel. "Multi-strand processes", involve parallel changes in "multi-strand relationships".

As an individual's competence grows the amount of stored information about each domain grows, extending the variety and complexity of situations they can cope with (e.g. predicting what will happen, deciding what to do to achieve a goal, understanding why something happens, preventing unwanted side-effects, reducing the difficulty of the task, etc.)

- **On-line vs Off-line intelligence**

Many animals can learn to manipulate objects, using on-line intelligence. A dog can learn to catch a thrown ball, a dolphin can learn to balance a ball on its nose, and many birds seem to be able to learn to build nests (e.g. a young male bowerbird tries copying nests built by an older male). The performance of such tasks uses "on-line intelligence" controlling actions either ballistically or using visual or proprioceptive or haptic servo-control. There are now many AI/Robotics research labs in which robots learn through repeated attempts with some sort of feedback from successes and failures to shape their behaviours to fit the requirements of behavioural task. This work usually assumes that states of the system, perceptual contents, actions, goal states, and in some cases rewards, can all be expressed as numbers or collections of numbers, as opposed, for example, to descriptions of relationships, e.g. "keeping the baby within my field of view" or "preventing the dog's lead wrapping round my legs".

Within this framework of behaviour-centred learning much interesting research has been done, and there have been many impressive advances that generalise what can be learnt or speed up what can be learnt, or make what has been learnt more robust.

But I want to raise the question whether this kind of research sheds much light on human intelligence or the intelligence of many other animals with which we can interact, or helps much with the long term practical goals of AI or explanatory goals of AI as the new science of mind. The main problem is all this online intelligence leaves out what can be called "off-line" intelligence, which involves a host of ways of doing something about possible actions other than performing the actions, for example thinking about "what would have happened if..." or explaining why something happened, or why something was not done, or teaching someone else to perform a task, or changing the environment so as to make an action easier, or safer, or more reliable. These abilities seem to be closely related to the abilities of humans to do mathematics, including for example discovering theorems and proofs in Euclidean geometry, which our ancestors must have done originally without any teachers, and without using the translation of geometry into arithmetic that is now required for geometrical theorems to be proved by computer (in most cases).

A subset of species, including young children and apparently some corvids seem to have the **additional** ability to think about and reason about actions that are possible but are not currently being performed. This can sometimes lead to the ability to reflect on what went wrong, and how faulty performance might be improved, or failure produced deliberately, and in some cases the ability to understand successes and failures of others, which can be important for teachers or trainers. For example, a mother (or 'aunt'?) elephant seeing a baby elephant struggling unsuccessfully to climb up the wall of a mud bath may realise that scraping some of the mud away in front of the baby will make an easier ramp for the baby to walk up, apparently using "counterfactual" reasoning, as required for a designer or planner. A monkey or ape may be able to work out that if a bush is between him and the alpha male when he approaches a female his action will not be detected.

For example, a child who has learnt to catch a fairly large ball may be able to think about what will happen if she does not open out her palms or fingers before the ball makes contact with her. And she may also be able to think about what will happen if she does not bring her fingers together immediately after the ball makes contact with her two open palms.

This uses "off-line" intelligence. More is said about this distinction in [Sloman 1982](#), [Sloman 1989](#), [Sloman 1996](#), [Sloman 2006](#) [Sloman 2011](#)

The differences between on-line and off-line intelligence are sometimes misconstrued, leading to poor theories of the functions of vision -- e.g. the theory that different neural streams are used for "where" vs "what" processing, and the theory of "mirror neurons", neither of which will be discussed further here. For more detail see ([Sloman 1982](#)) and the related papers below.

On-line and off-line intelligence are sometimes combined, e.g. when possible future contingencies are being considered during the performance of an action, or a partly successful action is not interrupted, but while it is continued the agent may be reflecting on what had previously gone wrong and how to prevent it in future.

Many complex actions, such as nest building, hunting intelligent prey, climbing a tree, eating a prickly pear while avoiding thorns (See Richard Byrne) or constructing a shelter or house require a mixture of on-line and off-line intelligence, often in parallel or alternating performances.

See also the comments about Karen Adolph's work on learning in infants and toddlers [below](#).

- **Transformation from learnt reusable patterns to a "proto-deductive" system, possibly including "Toddler Theorems".**

For some domains, after the information acquired (by animal, child, or adult exploring a new domain, or possible future robot) has reached a certain kind of complexity, powerful cognitive mechanisms somehow transform that information into a more systematic form so that there is a core of knowledge from which everything else learned about the domain can be derived, along with a great deal more -- so that the learner is then able to cope with novel situations. This requires something like the replacement of a collection of exemplars or re-usable patterns with a **proto-deductive system**. This term is not intended to imply that logic and logical deduction are used.

The main consequence is that the learner can now **work out** things that previously had to be learnt empirically, or picked up from teachers, etc. This means that the realm of competence is enormously expanded.

This requires the use of information structures of variable complexity composed of components that can be re-used in novel structures with (context-sensitive) compositional semantics -- one reason why internal languages had to evolve before languages used for communication.

N.B. This is totally different from building something like a Bayes Net storing learnt correlations and allowing probability inferences to be made.

Bayesian inference produces probabilities for various already known possibilities. What I am talking about allows new possibilities and impossibilities to be derived, but often without any associated probability information: if a polygon has three sides then its angles **must** add up to half a rotation.

Compare using a grammar to prove that certain sentences are possible and others impossible. That provides no probabilistic information. In fact a very high proportion of linguistic utterances had zero or close to zero probability before they were produced. But that does not prevent them being constructed if needed, or understood if constructed.

The same can be said about possible physical structures and processes. Before the first bicycle was constructed by a clever designer, the probability of it being constructed was approximately zero.

- **A conjecture about (some) toddler theorems (An idea still to be fleshed out.)**

In the case of logical reasoning it is possible to make discoveries about which classes of inference are valid by starting from examples, then generalising, then discovering (in ways that are not yet clear) that the generalisation cannot have counter-examples (e.g. by reasoning about "typical" instances that have all the relevant features).

For non-logical reasoning, e.g. reasoning about transformations of a set of topological or geometric relationships, similar processes of reasoning without performing physical actions can provide new knowledge of about possibilities and necessities.

Kenneth Craik, Philip Johnson-Laird and others have suggested that internal models can be used for making predictions about possible actions http://en.wikipedia.org/wiki/Mental_model However most of them fail to notice the differences between being able to work out "what will happen if X occurs" and being able to reason about about what is and is not *impossible*, or what else will *necessarily* occur if X occurs.

Examples of discovering what is *impossible* are discussed in <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/impossible.html>

- The learner discovers various ways of characterising structures and processes.
- Processes that alter a structure, or which modify a process (e.g. initiating, or terminating, or speeding up or slowing down, or changing direction of, some motion or rotation) can also be represented though that may require a more sophisticated and abstract form of representation.

- For purposes of performing similar actions in different contexts, schematic versions of the actions may be useful: e.g. if two opposed flat surfaces with an object between them move together, then their continued motion will be interrupted before they are in contact. This abstraction might be expressed in a form of representation used to control grasping in a wide variety of situations. See <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/grasping-grasping.html>
- Later, some learners discover that it is possible to select and evaluate plans for sequences of actions by combining such abstract representations, omitting the actual parameters required to instantiate the actions.
- This allows reasoning about future actions to be performed in the abstract, the result being a plan that can be executed by inserting the parameters.
- Alternatively a composite action may be performed, and because it was successful it may be recorded as a schematic composite action (a re-usable plan) with some of the details replaced by "gaps" to be filled whenever the plan is used. (This idea is an old one in the symbolic planning community -- e.g. Strips, Abstrips, etc.)
- Later the learner can discover that in addition to running an abstract plan by filling its gaps (instantiating its variables), the learner can run the plan schematically in different contexts and discover interactions: e.g. you can have all the conditions for grasping something yet the attempt to grasp fails because there is some additional object between the grasping surfaces that is larger than the object to be grasped. This can be discovered without actually performing the operation in a physical situation -- merely "running" a schematic simulation. It does not need to have any specific parameters for the sizes and distances. This is not to be confused with performing an inference using probabilities.

The key idea is that under some conditions it is possible to discover that properties of a schematic structure or schematic process are **invariant** -- i.e. the properties do not depend on the precise instantiation of the abstraction, though sometimes it is necessary to add previously unnoticed conditions (e.g. no larger object is between the grasping surfaces) for a generalisation to be true.

This idea will have to be fleshed out very differently for different domains of structures and processes, or for different sub-domains of rich domains -- e.g. Euclidean geometry, operations on the natural numbers. (See examples about counting [below](#).)

The kinds of discoveries discussed here are not empirical discoveries, but that does not mean that the reasoning processes are **infallible**. The history of mathematics (e.g. the work of Lakatos [below](#)) shows that even brilliant mathematicians can fail to notice special cases, or implicit assumptions. Nevertheless I think these ideas if fleshed out would support Kant's ideas about the nature of mathematical discoveries, as discoveries of synthetic necessary truths. (As far as I know, he did not notice that the discovery processes could be fallible.)

The ideas in this section are elaborations of some of the ideas in [Chappell and Sloman \(2007\)](#).

- **Alternative forms of representation**

I have argued in the past that there are alternative forms of representation that can be used for reasoning, and modelling causal interactions.

- <http://www.cs.bham.ac.uk/research/cogaff/04.html#200407>
Interactions between philosophy and AI: The role of intuition and non-logical reasoning in intelligence (1971)
- <http://www.cs.bham.ac.uk/research/projects/cogaff/81-95.html#43>
The primacy of non-communicative language (1979)
- <http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#glang>
Evolution of minds and languages. What evolved first and develops first in children: Languages for communicating, or languages for thinking (Generalised Languages: GLs)? (Work done with Jackie Chappell.)

- The kind of proto-deductive system a human toddler can produce -- or a squirrel, or orangutan or a nest-building bird -- seems unlikely to use the kinds of deduction logicians understand well, based on propositional and predicate calculus, so a major research problem is to investigate alternative forms of representation. Jackie Chappell and I have presented some draft ideas about requirements for those alternative forms of representation, used for perception, for planning, for plan-execution, for making predictions, for enabling internal explanations (e.g. how something happens, how something works).

This is deeply connected with a Kantian theory of causation. See our 2007 'WONAC' presentations <http://www.cs.bham.ac.uk/research/projects/cogaff/talks/wonac/>.

[Added 27 Oct 2011]

It is also connected with our discussion of "internal" precursors to the use of language for communication -- in pre-verbal humans, in pre-human ancestors and in other species. E.g. see [Sloman Talk52](#) on Evolution of minds and languages.

- Not much is currently known about the mechanisms that acquire and use the information initially, or how the transformations occur, or what the new forms of representation are, nor whether changes of architecture are also required. However in the case of language learning it is known that the transformation to a proto-deductive system (using a grammar/syntax) produces errors because natural languages (unlike Euclidean geometry, Newtonian mechanics, etc.) have many exceptions. Dealing with the exceptions obviously requires a further architectural change, which is a non-trivial process.

If we treat language learning as a special case of something more general, found also in pre-verbal children and in other species that can see, think, plan, predict, and control their actions sensibly, that may give us new clues as to the nature of language learning.

- A more detailed analysis than I can present here would subdivide the learning and developmental processes into far more distinct categories, concerned with different [domains of information](#), including:

- Spatial structures and processes perceived in the environment;
 - Spatial structures and processes created, changed, or manipulated by the perceiver;
 - Different collections of properties and relationships, including metrical, semi-metrical, topological, properties and relationships are involved in different domains.
 - Some kinds of processes involve not just physical changes, but also purposes, information, knowledge, attempts to achieve, successes and failures, and various kinds of learning. Perceiving, characterising, or thinking about such topics requires specific forms of representation and specific types of content to be represented: using **meta-semantic competences**, for representing and reasoning about things that themselves represent and reason.
 - An individual that applies such modes of reasoning to its own competences and uses of its competences, the individual can be said to be developing **auto-meta-semantic** competences.
-

- **Ontologically conservative and non-conservative transitions**

It may be useful to distinguish

- **Ontologically conservative transitions**
 These are reorganisations into deductive systems that do not extend the ontology previously available -- so the same forms of representation suffice, and no new **types** of entity are referred to, though new inferences may be possible because of the greater generality of the "axioms" (or their analogues) of the deductive system, compared with the previously acquired empirical knowledge.
(Example to be added)
- **Ontologically non-conservative ("ampliative") transitions** reorganisations that introduce new entities and new symbols (or new forms of representation) to refer to the new entities.
- **Somatic and exo-somatic ontologies/forms of representation**
 In some cases, the new entities may be postulated as hidden **parts** of the previously known types of entity, as happens in many theoretical advances in science, e.g. adding atomic theory to early physics and chemistry, then adding new kinds of sub-atomic particles, properties, relationships.

In other cases, the new entities postulated are not contained in the old ones, for example, when an organism that initially has sensory and motor signals and seeks regularities in recorded relationships, including co-occurrences and temporal transitions, later adds to the ontology additional objects that are not parts of the available signals but are postulated to exist in another space, which can have (possibly changing) projections into the sensory space. One extremely important example of this would be extending the ontology to include objects that exist independently of what the organism senses, and which can be sensed in different ways at different times. The former is a **somatic** ontology, the latter an **exosomatic** ontology.

An example, going from sensory information in a 2-D discrete retina to assumed continuously moving lines sampled by the retina, or even a 3-D structure (e.g. rotating wire-frame cube) projecting onto the retina, is discussed in

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/simplicity-ontology.html>

Ontologically non-conservative transitions refute the philosophical theory of concept empiricism (previously refuted by Immanuel Kant), and also demolish symbol-grounding theory, despite its popularity among researchers in AI and cognitive science.

They also defeat forms of data-mining that look for useful new concepts (or features) that are **defined** in terms of the pre-existing concepts or features used in presenting the data to be learnt from. (Some work by Stephen Muggleton, using Inductive Logic Programming may be an exception to this, if some of the concepts used to express new abduced hypotheses, are neither included in nor definable in terms of some initial subset of symbols.)

- **Ontologically potentially non-conservative ("abstractive") transitions**

Sometimes the extension of an ontology involves introducing a new type or relationship or operator that is an abstraction from previously used examples. For example, a mathematician who notices properties common to addition and multiplication can introduce the notion of a **group**, which is a collection of entities and a function from a pair of entities in the collection to an entity in the collection, where the function satisfies some conditions, e.g. it has an identity, an inverse and is associative, etc.

It is easy to see that integers (though not just positive integers) with addition, and also rational numbers, both form groups.

- **Ontology formation by abstraction**

Abstracting from a particular domain to introduce a new concept, like group, does not imply that any other instances of the concept exist. But that does not mean that the concept "group" is defined in terms of the cases from which it was abstracted.

That's because it is possible to discover later that some newly discovered mathematical structure is a group, e.g. a set of translations of 3-D structures, with composition as the group operator..

Many mathematical abstractions go beyond the exemplars that led to their discovery. In fact the discovery may be triggered by relatively simple cases that are much less interesting than cases discovered later. The initial cases that inspired the abstraction may be completely forgotten and perhaps not even mentioned in future teaching of mathematics.

This use of abstraction in mathematics is often confused with use of metaphor. Unlike use of abstraction, use of metaphor requires the original cases to be retained and constantly referred to when referring to new cases, whereas an abstraction can float free of the instances that triggered its discovery.

- There's much, much more to be said about all these topics. Some of these processes were modelled nearly 40 years ago by Gerry Sussman in his HACKER system, for his MIT PhD thesis, later published as a book. G.J. Sussman, **A computational model of skill acquisition**, American Elsevier, 1975, <http://dspace.mit.edu/handle/1721.1/6894> There's a useful summary of his work in Margaret Boden's 1978 book: **Artificial Intelligence and Natural Man**, Harvester Press, Second edition 1986. MIT Press,

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- The leading researcher into these processes, among psychologists and neuroscientists, seems to be Annette Karmiloff-Smith. I have a personal (and still incomplete) summary and review of her work here:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/beyond-modularity.html>

- As far as I know, the task of replicating such processes in robots is beyond the current state of the art in AI (except perhaps in 'toy' domains). We'll need to find new forms of representation, and new mechanisms for reorganising information in ways that produce powerful new ontologies and new representations. Perhaps this can build on the theory of construction-kits sketched in another document:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/construction-kits.html>

Some of the problems are discussed in more detail in

- <http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#talk86>
Talk 86: Supervenience and Causation in Virtual Machinery
- [Talk 111 \(below\)](#) on functions and evolution of language and vision.
- <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-theorem.html>
Hidden Depths of Triangle Qualia

And related documents referenced in those.

- <http://jackiechappell.com/news/tecwyn-anim-cogn-2011.html> Jackie Chappell, Cognitive strategies in orangutans (2011).

CONTENTS

- **HOW TO COLLECT DATA**

Many psychologists (in my experience, in several different universities and at conferences) have been educated to think that all scientific evidence must include numbers, correlations, and graphs.

That is a result of very bad philosophy of science. I'll outline some alternatives.

Much research on children (and other animals) is restricted to looking at patterns of responses to some experimenter-devised situation. This is like trying to do zoology or botany only by looking in your own garden, or doing chemistry only by looking in your own kitchen. It is based on a failure to appreciate that many of the most important advances in science come from discovering what is **possible**, i.e. what **can occur**, as opposed to discovering laws and correlations. This is explained in more detail in Chapter 2 of **The Computer Revolution in Philosophy** (1978)

<http://www.cs.bham.ac.uk/research/projects/cogaff/crp/chap2.html>

How to discover relevant possibilities: First try to find situations where you can watch infants, toddlers, or older children play, interact with toys, machines, furniture, clothing, doors, door-handles, tools, eating utensils, sand, water, mud, plasticine or anything else.

Similar observations of other animals can be useful, though for non-domesticated animals it can be very difficult to find examples of varied and natural forms of behaviour. TV documentaries available on Cable Television and the like are a rich source, but it is not always possible to tell when scenarios are faked.

Some videos that I use to present examples are here:

<http://www.cs.bham.ac.uk/research/projects/cogaff/movies/vid> More examples are presented or referenced below. Some are still in need of development: more empirical detail and more theoretical analysis of possible mechanisms.

This discussion of explanations of possibilities is also relevant:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/explaining-possibility.html>

[To be continued.]

- **HOW TO THINK ABOUT WHAT YOU OBSERVE**

Doing science requires formulating deep questions, and, if possible, good answers. Without good questions it's unlikely that the answers will turn up. Many of the research questions commonly investigated are very shallow:

Which animals can do X?

At what age can a human child first do X?

What proportions of children at ages N1, N2, N3, ... can do X?

Under what conditions will doing X happen earlier?

What features of the situation make it more likely that a child, or animal, will do X?

Which aspects of ability, or behaviour, or temperament are innate?

To avoid shallow questions, learn to think like a designer:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/design-based-approach.html>

Sometimes that requires thinking like a mathematician, as illustrated below in several examples -- a designer needs to be able to reason about the consequences of various design options, in a way that covers non-trivial **classes** of cases (as opposed to having to consider every instance separately).

That often involves discovering, and reasoning about, invariants of a class of cases. For example, an invariant can be a feature of a diagram that supports reasoning about all possible circles or all possible triangles, in Euclidean geometry. Usually that does not require the diagram to be accurate.

When children are taught to measure angles of a collection of triangles to check the sums of the angles, they are NOT being taught to think like a mathematician.

Sometimes people who are not able think like a designer or a mathematician resort to doing experiments (often on very small and unrepresentative groups of subjects). I have compared that with doing Alchemy, here:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/alchemy/>

(Is education research a form of alchemy?)

Unfortunately, the educational experience of many researchers includes neither learning to think like a mathematician nor learning to think like a designer.

E.g. many people who can state Pythagoras' theorem, or the triangle sum theorem have no idea how to prove either, and in some cases don't even know that proofs exist, as opposed to empirical evidence obtained by measuring angles, areas, etc.

[Note]

A sustained onslaught against bad science and bad philosophy can be found in Chapter 12 of David Deutsch *The Beginnings of Infinity: Explanations that transform the world*. I think his criticisms apply to much psychological and neural research on mathematical competences in humans and other species -- done by scientists who would not know how to give a robot such competences.

[To be continued.]

● **Some Common Types of Erroneous Thinking**

(In researchers, not their subjects!)

○ **Unitary scalar metrics vs partial orderings and semi-metrical orderings**

It is often assumed that an intelligent agent must use unitary scalar notions of length, angle, orientation, size, weight, etc., as part of a global cartesian framework for geometry. For example, it is commonplace for a robot vision system to have to go through a calibration process when it is switched on.

I suspect that most animals never achieve use of a global euclidean ontology with global metrics, but that does not stop them seeing things and using vision to select goals and plans and to control their movements, and predict movements of others.

I also suspect that in humans the uses of global metrics and coordinate frames result from long periods of using something more primitive, and that it requires a special education that was not available to our ancestors thousands of years ago to be able to think of all lengths (angles, areas, volumes, speeds, etc.) as comparable using a common metric for each quantity. But even without that education toddlers are very effective in coping with most of their normal environments. How is that possible?

Instead of global coordinate systems, perhaps they use less precise and general, but somewhat more complex ontologies based on use of networks of partial orderings augmented with semi-metrical extensions, which use the fact that even without global metrics, it is possible for differences (e.g. in length, angle or area) to be compared, even when absolute values are not available. E.g. The pine tree is taller than the lamp-post by an amount that is greater than the height of the lamp post, but less than the height of the tree between them.

○ **Uncertainties vs probabilities**

It is often assumed that uncertainties need to be represented as probabilities.

I suspect that is a deep error, and that for many biological organisms instead of probabilities the ontology includes

- Possibilities (collections of possible actions, states, processes, consequences, etc.)

- Impossibilities: types of combinations of states, events, or processes that cannot occur, e.g. increasing the height of water in a cylindrical container while decreasing the volume.
- Comparisons within possibility-collections, discussed in the next section.
(Added 17 Nov 2020)

○ **Comparisons of likelihoods**

A partial ordering of possibilities as being more or less likely (sometimes expressed, confusingly, as more or less possible). This colloquial usage should not be confused with the technical term "likelihood" which implies a numerical value.

I.e. if P1 is the possibility of an agent A moving with heading H colliding with the door frame, and P2 is the possibility of A passing through the doorway without collision, A may know that H makes P2 more likely than P1, which is why the heading H was selected. (For more on this sort of reasoning see "[Predicting affordance changes](#)".)

However if P3 is the possibility of disliking the food available to A at the next feeding opportunity, there may be no basis for deciding whether P3 is more or less likely than either of P1 or P2. The heading H, which affects the relative likelihood of P1 and P2 will normally be considered irrelevant to P3, even though there may be a theoretical connection, e.g. if A gets seriously injured colliding with the door frame, there may be medical restrictions on food offered during a recovery process. This information may not be available to A, and even if it is available it need not be sufficient to derive a likelihood ordering. Even the most knowledgeable scientist may be incapable of doing that, mainly because the question which is more likely has indeterminate semantic content, since so many different possible but unspecified contexts can affect the comparison.

One aspect of intelligence is the ability to think of contexts that affect the relative likelihoods of possibilities under consideration: that is also a key component of mathematical and engineering design competence.

For discussion of non-metrical aspects of perception of affordances (possibilities and impossibilities instead of probabilities, and use of partial orderings instead of scalar measurements) see

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/changing-affordances.html>
Predicting Affordance Changes
(Steps towards knowledge-based visual servoing)

○ **Bayesian/causal/predictive nets vs equivalence classes and partial orderings**

It is often assumed that learners collect empirical information into probabilities in Bayesian nets. Such nets can be used to derive predictions ordered by probability.

I suspect that what actually goes on in learners, which is misinterpreted by the Bayesian theorists, is much more subtle and much closer to discoveries of useful equivalence classes, e.g. concerning which a form of mathematical reasoning can be used. When we find out how to give machines ways of constructing those equivalence classes and ways of reasoning about them, our robots will be far more intelligent and human-like -- or animal-like -- than they are now. NB: I am not an expert on Bayesian mechanisms and may have misunderstandings and gaps in my

knowledge.

A simple example: a child can count, and can turn a coin over while counting, may discover a relationship between the starting position (heads or tails up), the number of turns and the final position. Initially that may be an empirical discovery, and may even be expressed probabilistically if the child makes counting errors. But later on the child will be able to work out what the result **must** be on the basis of whether the number of turns is odd or even. There are more complex examples below. (That's too vague: this is work in progress.)

- **EXAMPLES:**

- Domains for toddler theorems
(and some post-toddler theorems)**

- These examples provide fragmentary evidence for the diversity of domains of expertise and the kinds of knowledge transformations they make possible.

- Some of the examples illustrate portions of the process of information re-organisation (perhaps instances of what Karmiloff-Smith means by "Representational Redescription"?).

- The [list of examples](#) in this document is a tiny sample. I shall go on extending it. (Contributions welcome.)

- Some of the examples were inspired by the wonderful little book [Sauvy and Sauvy \(1974\)](#).

- NOTE:**

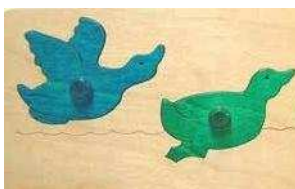
- The order of the examples presented here is provisional. Later I'll try to extend the list and impose a more helpful structure.

Examples of Use of Knowledge About Physical Objects

- **Problems of alignment when manipulating and stacking objects**

- At first very young children playing with 'lift out' toys like these find it difficult to insert a cut-out picture into its recess, even if they remember which recess it came from.

- E.g. They put the picture down in approximately the right place and if it doesn't go in they may press hard, but not attempt any motion parallel to the picture surface.



- After a while they seem to learn that both the recesses and movable objects have boundaries, and that when flat objects are brought together the boundaries may or may not be merged.

- At this more advanced stage, a child may place the picture object in roughly the right place and then try sliding and rotating until it falls into the recess.

Still later, the child realises that boundaries can be divided into segments and that segment of the recess boundary may match a segment of the object boundary, and then try to insert the object by first ensuring that matching segments are adjacent and then slightly varying the location and orientation of the piece until it falls into the recess.

Long before they can do this, I suspect they can insert a circular disc into a recess, since there is no problem of alignment. If there are different discs and recesses of different sizes the insertion requires size and location to be perceived and used in controlling the insertion process. When the items are not symmetrical, inserting requires

- a) identification of matching portions of the recess and the movable piece,
- b) the ability to match locations and orientations of the two boundaries,
- c) depending on how tight the fit is, the ability perform slight movements to compensate for imprecision in the placing action,
- d) in some cases using a tilted insertion orientation to allow the shape of the recess to guide the inserted piece into the exact location and orientation.

There are similar problems stacking cups, except that in addition to the shape of boundary, the size can be very important, and children may have to learn to order the sizes in order to ensure that all the cups can be stacked. There are probably many intermediate discoveries that can be made and used, some of them red-herrings because they only work by accident in certain conditions, or because they allow a cup to be stacked but prevent ALL cups being stacked, e.g. placing the smallest cup on or in the largest cup.



○ **Sorting or stacking objects by height or size**

See the short, tentative, discussion in this PDF presentation:

<http://www.cs.bham.ac.uk/research/projects/cogaff/talks/math-order-stacking-sloman.pdf>

[Fiona McNeill](#) provided this example of a domain still being explored and only partially understood by the child, in March 2009:

"One interesting aspect of Eilidh's ontology that I noticed over the weekend:

She has stacking cups that go inside one another that she loves to play with. Until recently, getting them to go in the right order was more or less a case of trial and error, but she has just made a big step forward.

She is now very good at noticing 'holes' - so if she has, say, cups 2,3,5,6 all stacked, and 1,4,7 loose, she will immediately remove cups 2 and 3, recognising with no apparent effort that something needs to go between them and the bigger cups 5 and 6.

However, she seems to have no concept of relative size and will, seemingly, pick up either 1, 4 or 7 with equal probability to put them in this hole, not perceiving that 7 is clearly too big to go into 5 or that 1 is clearly too small to fill the hole.

I would have thought that judging relative size, when there is a fairly large difference in the sizes, would be far more instinctive than noticing that cup 3 is a little loose in cup 5, which is not immediately obvious to the eye. Apparently not!

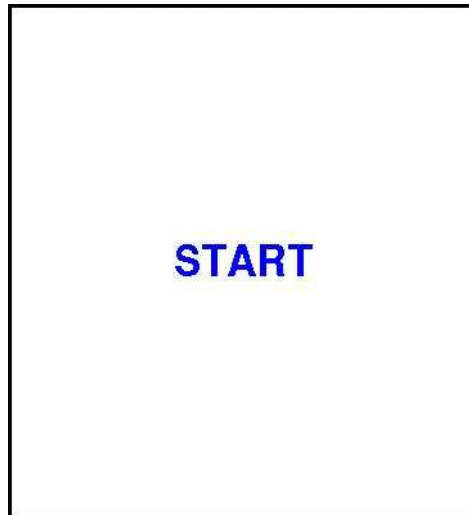
She has also does not have the concept of 'largest object'. If she starts off by picking up the biggest cup (cup 10), she will try to fit it into all the others, and when it will not, instead of trying to fit something into it, she tries again and again to fit it into another one, getting increasingly frustrated. I usually put it down for her and put another in it, and then she is happy to go on putting cups into it, but she has not got this for herself yet."

NOTE: there is research that shows surprising insensitivity to size differences in young children. [DeLoache et al. \(2004\)](#) state:

18- to 30-month-old children sometimes fail to use information about object size and make serious attempts to perform impossible actions on miniature objects. They try, for example, to sit in a dollhouse chair or to get into a small toy car. We interpret scale errors as reflecting problems with inhibitory control and with the integration of visual information for perception and action.

○ Exploring topology/holes

This toddler (age about 17.5 months) seems to be exploring topology. She spontaneously crawled towards the sheet of card while holding a pencil, picked up the card, pushed the pencil through the hole, pulled the pencil out, moved the pencil up and over the edge of the card while rotating the card toward the pencil then pushed the pencil through the hole from the opposite side, then removed the pencil, reverted to the original side and finally pushed the pencil in then pulled it out again.



Note: this 'gif video' may not work for you in this context. It can also be viewed in this video, which includes a commentary and some slow motion: [small-pencil-vid.webm](#)
(Old video replaced 19 Aug 2017)

All this was done with intense concentration, and apparently ignoring other people in the room. There was no attempt to communicate what she had discovered, or to seek approval for her achievement. The video may be more revealing slowed down (e.g. using vlc), so that relationships between posture, direction of gaze, how the objects are held, how they are moved, etc. can all be taken in. This was a first attempt, with no trial and error learning required.

This appears to be a case of "architecture-based" motivation discussed [above](#). There is no need for such behaviour to be generated by anticipation of any kind of reward, although in special cases it could be. But this child seemed to be merely reacting to opportunities in her environment. There were adults and an older child in the room but the toddler seemed not to be paying attention to any of them, and certainly did not appear to be seeking signs of approval during or after her performance.

NOTE: Manipulating the pencil and card, and getting the pencil into the right position and orientation to push it through the card from each side would be a significant challenge for a robot. There is no evidence that she had previously been practising this action with a pencil and a hole in a card, though of course she had pushed other objects through holes, in very different physical circumstances. Note that when moving the pencil over towards the second side of the card she does not even look at the pencil, as she is peering over the card at the 'other' side of the hole. Yet she not only moves the pencil toward the new side of the card, while doing so she also automatically rotates it into the required new orientation. This seems to suggest a good grasp of the 3-D structure of the space she is in and how to move things around in space to achieve some of her goals. Her grasp of space is not perfect as she sometimes has trouble rotating 3D objects into the right orientation to fit through a hole, e.g. rotating a triangular prism to fit through a triangular hole.

This child's ability to talk was still very limited: she could produce some very short sentences in understandable English, and could understand more. However it seems clear that she had complex intentions that her actions were designed to achieve that were beyond her spoken linguistic capabilities, e.g. getting the point of the pencil to the hole, on three different occasions, rotating the card until she can see the hole from the other side, getting the point of the pencil

through the hole from that side, removing the pencil, etc. It is very unlikely that those goals could have been expressed in terms of the required sensory and motor signals -- that level of detail would be far too specific: she must have had some more abstract internal **language** for specifying a state of affairs, which she could use both in order to bring about that state of affairs (by deriving control processes from the specification of the goal) and to check whether it had been achieved, so that a new task could be adopted. There is no reason to suspect that the intended actions were planned in full metrical detail in advance -- an alternative form of representation using partial orderings is sketched in

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/changing-affordances.html>

Predicting Affordance Changes

(Steps towards knowledge-based visual servoing)

I think little or nothing is known about the role of a toddler brain in these processes.

Of course, similar comments can be made about many other intelligent animals that do not show any sign of using human languages, including nest-building birds, squirrels defeating squirrel-proof bird feeders, parrots able to rotate a nut to a desired orientation by alternatively holding it in beak and foot, hunting mammals bringing down prey, and then extracting food from the interior, and many more. For a discussion of issues related to evolution of vision and language and conjectures about precursors of human language see this presentation: [Talk111](#).

- **Toddler theorems about walking, falling backwards, and trampolining**
(This section should be expanded and split into smaller sections).

Walking

(To be added)

Falling backwards

(Reported by [Michael Zillich](#) April 2009. Name of toddler changed.)

"LLLL last week suddenly learned to walk. It seems she figured that handling her little suitcase while crawling was too cumbersome and so just stood up and walked, carrying the suitcase around for hours :)

Now she also walks on quite uneven ground outside.

One really nice detail: She is quite good at maintaining balance (briefly stopping to regain it when necessary) and at using her hands (and bottom) to cushion falls, in case balance is truly lost.

But when she is in our bed, with soft cushions and blankets, she loves to stand up straight and simply let herself fall backwards, with a relaxed sigh. She knows she can only do this in bed. We did not teach or show to her (I am too tall to do that) so she had to figure that out herself. And she seems to enjoy the "thrill" of losing control."

Playing on a trampoline

I noticed a young (probably pre-verbal) child playing on a trampoline. He had discovered that he could jump up, stick his legs forward and fall so that his bottom hit the trampoline -- without hurting himself. I presume he would not attempt such an action on a hard floor. It's unlikely that he is able to express verbally the assumptions

he is using about the physical properties of the trampoline, but he is using them nevertheless.

It's very hard to find out what such a child does and does not understand by asking questions, though Piaget tried hard (see his last two books, on Possibility and Necessity, translated 1987).

One could (though probably should not) invite the child to try the same action on different surfaces, e.g. a lawn, a hard floor with a thick or thin carpet, a bed on which he is standing, a sandpit, etc.

If the child is old enough to discuss such possibilities probing questions may or may not reveal the stage of theory development (as opposed to skill development). If it's too early for verbal interrogation there may be no substitute for long term observations of spontaneous actions in a playground, perhaps a special purpose playground with different surfaces (and close supervision).

Such research should not be corrupted by spurious requirements to collect statistics about what happens when. It's what **can** happen, that's important for deep science, and how those possibilities emerge, and how they are constrained. (Piaget understood this, but many of his critics did not.)

Three children on a trampoline

I watched three children on a trampoline. The youngest seem to be pre-verbal though he could walk and climb. The oldest was a boy who might have been four or five years old. In between, was a girl who seemed to be at an intermediate age (and size).

At one stage the girl started going head over heels on the trampoline: jumping in such a way that her hands and head hit the trampoline with her trunk going over. The other two were intrigued.

The little one seemed to want to do something inspired by her tumbling, but did not seem to know what to do. He jumped around a bit stepping with alternate feet on the trampoline then seemed to give up.

The older boy seemed to know that he had to do something about getting his head down, but at first merely made clumsy and ineffectual movements. (I wish I had had a video recorder.) After a few attempts he seemed to realise what was necessary, and managed to go head over heels several times, rather clumsily at first and then apparently with greater understanding of the combination of movements needed to initiate the tumble, after which momentum and gravity could complete the process.

I don't think any of them could express in a human communicative language what they had learnt but clearly there was something in the information structures they created internally, to function as a goal specification, as a control strategy for actions to achieve the goal, as a critical evaluation of early attempts, as a debugging process to modify the details of the action so as to complete and "clean up" the final desired action.

Modelling this on a robot (possibly simulated -- to reduce the risk of damaging expensive equipment!) would not be trivial. The process involves a mixture of fine control with ballistic action and requires sufficient understanding to manage the initial controlled movements in

such a way as to launch the right kind of ballistic action.

It does not seem to me that these children are making use of something like the standard statistical AI approach to learning which requires a space of motor (or sensory-motor) signals to be sampled using statistics (and perhaps hill-climbing) to direct the search, possibly using a numerical evaluation/reward function. I suspect they are using richer and more varied information structures in a complex self-improving control architecture.

Karen Adolph's work is also relevant:[Adolph \(2005\)](#)

It is important to distinguish the acquisition of

- 'on-line' intelligence, investigated by Adolph, which involves learning to control actions as they are being performed (e.g. catching a ball, falling in a way that prevents injury) and
- 'off-line' or 'deliberative' intelligence which involves being able to represent and reason about **classes** of processes, including some of their invariant properties -- discovered in toddler theorems and later on in more sophisticated theories.

Various kinds of deliberative competence are discussed in [\(Sloman 2006\)](#)

Riding the back of a sofa

(Added 12 Dec 2013)

Bob Durrant provided this example.

"To add to your list of toddler theorems - my three year old daughter has learnt, by unguided exploration as far as I can tell, that:

She can straddle the back of the sofa without toppling it over.

Facing right w.r.t the front of the sofa, if she wants to get down from the sofa to the rear of it from a straddling position, then (using her right hand to support herself) she rotates clockwise on her bottom about 90 degrees to bring her left leg over and slides off, using her bottom as a brake to control rate of descent, to land standing up.

If she wants to get off the other way then she either does as above (with her right leg and left arm) to end up standing on the cushions or, because it is more fun, she instead lifts over her left leg and she tumbles backwards on to the cushions.

She has never, as far as I know, tried to tumble the other way (i.e. over the back of the sofa, with a fall of about twice as far on to the carpeted floor).

Prior to this she did similar from the arms of the sofa and armchair, again without ever (intentionally) tumbling the wrong way as far as I am aware."

What are the implied toddler theorems at work here?

Playing on a slide

-- trying to throw a teddy-bear to a child at the top of the slide.

-- walking up a slope while holding onto a rope attached near the top.

[To be continued]

○ **Problems of moving objects in a complex structured environment.**

See some of the videos here, especially the child pushing a broom (video 6):

<http://www.cs.bham.ac.uk/research/projects/cogaff/movies/vid>

This is an example of **matter manipulation**, a type of competence that subsumes tool-use and many other things that have been studied in children and other animals.

A broom can be thought of as a "tool for shifting dirt on a floor", but in the video is not being used in that way. Rather the child appears to be moving the broom around for its own sake, rather than for the sake of some other effect.

Such matter-manipulation sometimes has utilitarian functions (e.g. obtaining food, putting on clothes, getting hold of some object that is out of reach) but need not have. With or without serving an explicit goal of the manipulator the processes seem to be a pervasive type of activity in very young children and also some other animals.

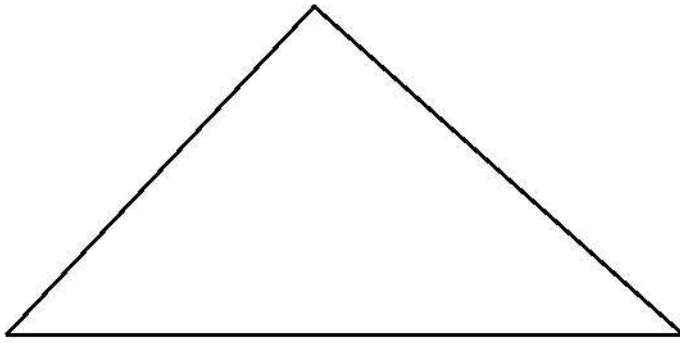
Presumably this is because playful, exploratory, manipulation can provide much information about, for example:

- What portions of objects can interact when objects are moved, or move spontaneously, and what are the consequences of those interactions? (many types of surface fragment coming together, coming apart, sliding, pushing, being obstructed by, guiding, and many more).
- What kinds of surface fragments objects can have, e.g. corners, edges, curves of various sorts (convex, concave, saddles), holes, cracks, gaps,
- How the relative movements of objects can be constrained in various ways, e.g. by shapes of surfaces, by glue, pins, hinges, grooves, gears, strings, wrappings, etc. ... many more...
- What kinds of information can be obtained or obstructed by manipulating objects e.g. -- about the properties of different kinds of matter -- about how to get different sorts of information by moving in or changing the environment (e.g. opening a container, looking through a hole or window, moving closer to a door, moving away from the door). -- about the reactions of other animate entities (e.g. siblings, parents), etc.
-

CONTENTS

Geometrical and other reasoning about what is and is not possible

- **Learning to think about changes that could happen but are not happening.**
EXAMPLE: Thinking about triangles. Consider an arbitrary triangle



Suppose it is formed from a stretched rubber band held in place by pins.

There are many ways the shape, size, orientation and location of the triangle could be transformed, by moving the pins.

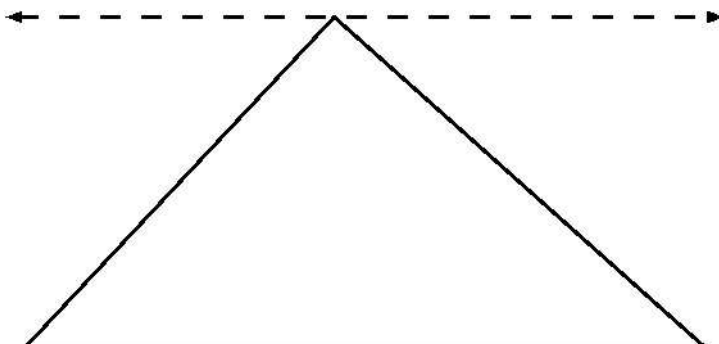
Think of some possible changes do-able by moving one, or two or all three pins, and for each change try to work out its consequences.

That is an easy task for a mathematician since much of mathematics is a result of the human (animal?) ability to look at something and think about how it could be changed, and what the consequences would be.

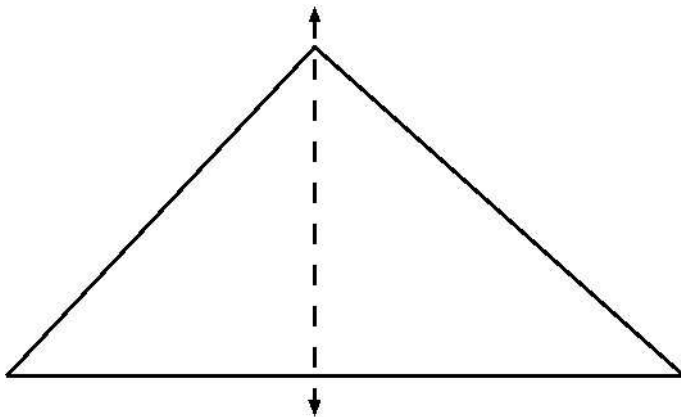
Most humans do it often in everyday life, e.g. when considering rearrangements of furniture.

The ability to do this develops slowly and erratically in children -- and in cultures! See also (Piaget & others, 1981, 1983)

Among the many possible ways you could alter the triangle, e.g. moving, or rotating the whole thing there is one that involves moving only one pin, **parallel** to the opposite side, in either direction, e.g. moving the top pin here, parallel to the opposite side (the "base").



Another possibility involves moving the top pin up or down in either direction **perpendicular** to the opposite side.

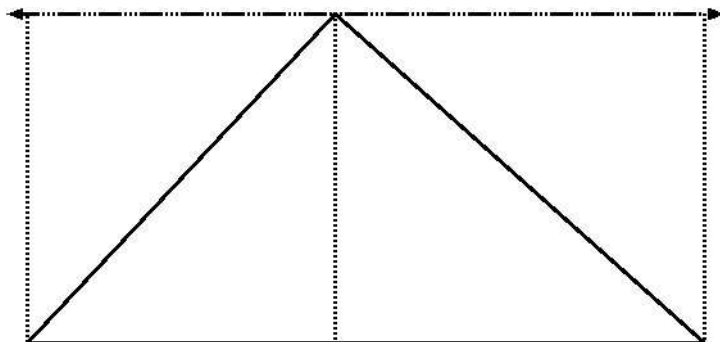


Can you see any interesting difference between those two sets of possible changes to the configuration?

One set of changes will increase or decrease the total area of the interior of the triangle.

The other set of changes will leave the area of the triangle unchanged.

Can you see why that must be so? Here's the explanation:



If you don't recognize what's going on, try reading this introduction to thinking about triangles and their areas:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-theorem.html>

The crucial point about such a diagram is that (like all diagrams used in proofs in Euclidean geometry) the relationships perceived in the diagram do not depend on the specific size, shape, colour, location, orientation, etc.

They don't even depend on the diagram being drawn accurately (with perfectly thin, perfectly straight lines). That's because once the proof is understood correctly its scope covers a very large class of abstraction. It's not clear that people not trained in mathematics can easily think that way.

There's an interesting 'bug' in the proof-sketch as shown in the diagram which is related to the need to do proper case analysis. It's a simple example of the sort of phenomenon discussed by Imre Lakatos in **Proofs and Refutations**, mentioned below. The bug in the 'chocolate' theorem, discussed below, is another example. Identifying the bug is, for now, left as an exercise for the

reader, though mathematicians will find it obvious.

Max Wertheimer discussed an analogous bug in a proof given by a school teacher regarding the area of a parallelogram, described in his book **Productive Thinking**. More examples of buggy, but fixable, proofs are given below.

[The relationship between this sort of bug and the problems a child has in handling exceptions to grammatical rules in language may be illuminating, as regards information processing architectures and mechanisms required.]

This human ability to reason about necessary consequences of alterations to configurations in the environment may be closely related to Kenneth Craik's hypothesis that some animals can use internal models of the environment to work out consequences of possible actions. (Craik, 1943)

Compare also (Karmiloff-Smith, 1992), and Piaget's work on possibility and necessity, and also Kant's philosophy of mathematics (Kant 1781).

Work that remains to be done includes finding out how a child, or non-human animal, or future robot, could notice that some collection of structures and processes forms a **domain** that has interesting properties, including invariants that are discoverable by reasoning about the structures and relationships, how the relationships can be discovered and supported by a non-empirical argument, how different domains can be combined to form new domains of expertise, and how all of this can lead to the phenomena of Representational Redescription discussed by K-S.

We also still need to understand how to get robots and other learning machines to go through similar procedures. See also:

<http://www.cs.bham.ac.uk/research/cogaff/96-99.html#15>

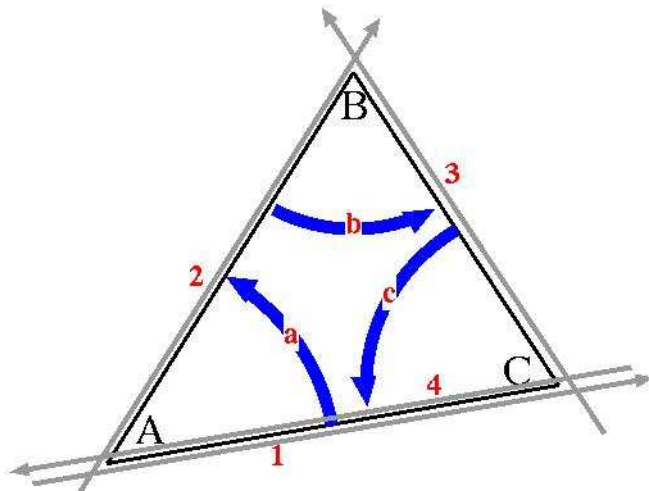
A. Sloman, Actual Possibilities, in
Principles of Knowledge Representation and Reasoning:
Proc. 5th Int. Conf. (KR '96),
Eds. L.C. Aiello and S.C. Shapiro,
Morgan Kaufmann, Boston, MA, 1996, pp. 627--638,
Added 11 Sep 2013

○ **Discovering the triangle sum theorem**

For an application of the ideas above, to formalising notions of space and process, see

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/p-geometry.html>

Based partly on ideas by Mary Pardoe developed while she was teaching children mathematics. Here's an extract from that discussion:



Many years ago, Mary Pardoe, then a school-teacher, discovered this excellent proof of the theorem that the internal angles of a triangle on a plane surface add up to a straight line:

Consider moving the arrow from position 1 (going through A to C) through the three rotations labelled **a** to **c**.

The arrow must end up at location 4 (going through C to A)

It has turned through each of the internal angles A, B, C.

What is the total rotation of the arrow?

Does this demonstration depend on size, shape or location of the original triangle?

A variant of this proof that uses not only rotations, but also translations, is:

<http://www.cs.bham.ac.uk/~axs/fig/angles-segment-proof.jpg>

A discussion of the assumptions is here:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/p-geometry.html>

Compare:

Roger B. Nelsen,

Proofs without words: Exercises in Visual Thinking,

Mathematical Association of America, 1993

(Freely available on google books: books.google.com)

I first met Mary when she was a student (Mary Ensor) at Sussex University, where I was a lecturer.

For more information contact me using the email address on my web site: <http://www.cs.bham.ac.uk/~axs/>

Note: Like the "standard proof" using parallel lines, this one cannot be used on a curved surface

Aaron Sloman - 23rd Dec 2010

ADDED 10 Sep 2012, Updated 9 Apr 2013:

A more detailed analysis of requirements for discovering theorems in geometry is:

"Hidden Depths of Triangle Qualia"

<http://tinyurl.com/BhamCog/misc/triangle-theorem.html>

○ **Discovering what can and cannot be done with rubber bands and pins**

If you have a rubber band (elastic band), some pins, and a board into which the pins can be stuck, you can make figures by using the pins to hold the band stretched into a shape bounded by straight lines (if the band is stretched between the pins).

The following are sample questions about what is possible, what is impossible, and how many pins or rubber bands are needed to make something possible.

For example, you can make a triangle, a square, an outline capital "T" with one rubber band and a set of pins?

Is it possible to make an outline capital "A" ?

Is it possible to make a circle?

Is it possible to make a star-shaped figure, with alternating convex and concave corners?

What's the minimum number of pins required for that?

How can you be sure?

For more examples see

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/rubber-bands.html>

<http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#toddler>

Learning about structures and processes related to kinds of matter

- **Discovering what can and cannot be done with various kinds of food** Mealtimes are a great occasion for exploring new domains involving foods of many kinds, the containers and utensils provided, and of course much social interaction in which things move under the control of adults suggesting a host of experiments for finding out more.

Here's a video of a child feeding yogurt to his belly, his thigh and a carpet, and doing several kinds of experiment with yogurt and spoon, presumably feeding his mind, though he probably does not know that:

<http://www.cs.bham.ac.uk/research/projects/cosy/conferences/mofm-paris-07/sloman/vid/yogurt-experiments-10mths.mpg>

There are more videos with very short comments that need to be expanded, here:

<http://www.cs.bham.ac.uk/research/projects/cosy/conferences/mofm-paris-07/sloman/vid/>

For a PDF presentation on learning about different kinds of 'stuff' see

<http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#brown>

From "baby stuff" to the world of adult science: Developmental AI from a Kantian viewpoint. (Talk at Brown University 2009)

Some intermediate stages in development of competence/expertise/understanding

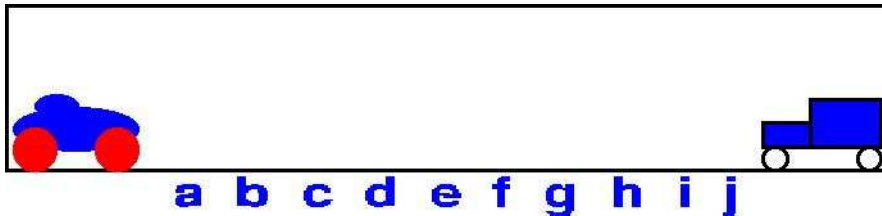
- **Evidence for partial construction of a theory for a domain.** Sometimes the process of construction of a new generative theory is in an intermediate stage, where the theory generates new answers to questions or problems, or new plans, but doesn't get things quite right. This happened once when a child was trying to persuade me that we should go on a picnic in mid-winter. When I objected that it would be much too cold in the middle of winter he responded:

"Today might be much more hotter than it usually bees"

More generally, the phenomena of "U-shaped" language learning provide many clues as to what goes on when information fragments acquired empirically are transformed into a "deductive" system, when the system needs to be capable of handling exceptions -- unlike the systems of topology, geometry, and other kinds of proto-mathematical knowledge.

○ **Five year old spatial reasoning: Partial understanding of motions and speeds**

Consider a slow moving van and a fast moving racing car. They start moving towards each other at the same time.



The racing car on the left moves much faster than the van on the right: Whereabouts will they meet -- more to the left or to the right, or in the middle?

One five year old answered by pointing to a location on the left, somewhere near "b" or "c".

Me: **Why?**

Child: **It's going faster so it will get there sooner.**

What produces this answer? Could it be:

- Missing knowledge?
- Inappropriate representations?
- Missing information-processing procedures?
- An inadequate information-processing architecture?
- Inappropriate control mechanisms in the architecture?
- A buggy mechanism for simulating objects moving at different speeds?

○ **Partly integrated competences in a five year old**

The strange answer to the racing car question can perhaps be explained on the assumption that the child had acquired some competences but had not yet learnt the constraints on their combination.

Here are some fragments that may have been learnt, but perhaps without all their conditions for applicability fully articulated.

- If two objects in a race start moving at the same time to the same target, the faster one will get there first
- Arriving earlier implies travelling for a shorter time.
- The shorter the time of travel, the shorter the distance traversed
- So the racing car will travel a shorter distance!

The first premiss is a buggy generalisation: it does not allow for different kinds of "race".

The others have conditions of applicability that need to be checked.

Perhaps the child had not taken in the fact that the problem required the racing car and the van to be travelling for the same length of time, or had not remembered to make use of that information.

Perhaps the child had the information (as could be tested by probing), but lacked the information-processing architecture required to make full and consistent use of it, and to control the derivation of consequences properly?

○ **Unanswered deep questions**

- What forms of representation are available for the child for recording and using such information?
- What sorts of mechanisms or algorithms are available for making use of the information?
- What is the whole architecture that can acquire, store, grow, transform and use such information?

Is Vygotsky's work relevant?

Some parts of Piaget's theory of "formal operations"?

Compare Karmiloff-Smith on "Representational Redescription", discussed in <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/beyond-modularity>

Could the child's reasoning be evidence for a process of representational redescription that is still incomplete: i.e. generally useful items of information that can be recombined in different contexts have been extracted from the collection of empirically learnt associations. But the conditions for recombination, and the constraints on applicability of inferences, have not yet been discovered. In principle, this looks like a type of learning that could be modelled in terms of construction of a rule-set capable of supporting deductive inference.

(I think Richard Young's PhD thesis around 1972 was concerned with a process something like this, but involving ordering of objects by height.)

○ **A two year old Aristotle? (Added 7 Aug 2013)**

An example that may or may not indicate partial understanding comes from two year old Ada, daughter of Dov Stekel and Diane Levine, reported with their permission.

Today, our daughter Ada (named for Lovelace), who turned 2 earlier this month, said "Kitties have tails. I do not have a tail. I'm not a Kitty."

Is it possible that a two year old has grasped the general principle that from premises

Xs have Ys
A doesn't have Y

it follows that

A is not an X ?

Some initial thoughts about this:

This may be an indication of a stage in which something less is understood than the words suggest. Finding out exactly what a child does and does not understand may require careful probing (as opposed to use of standardised tests).

It is possible that children start noticing patterns of related truths and only later, as a result of some form of "representational redescription" (see [this discussion](#) of Annette Karmiloff-Smith's work), grasp the general principles.

During the transition various partial competences may be displayed. This is consistent with the theory of "meta-configured" competences in [Chappell & Sloman 2007](#).

Ada may be a highly precocious and unusual logician. Another option is that she was not making an inference, merely noticing that there's an important structural relationship between the three assertions. On Karmiloff-Smith's theory, learners can develop a high level of empirical competence before they do the structural reorganisation that allows old generalisations to become 'theorems' (not her word) along with many that become derivable only after the old information structures are replaced with new "generative" forms.

Investigating a young mind is a very difficult thing to do. Non-performance in tests generally proves nothing at all, and even successful performance can be hard to interpret.

We could try delicately, and tactfully, probing, by finding a way to introduce structurally similar new examples to see if she draws similar conclusions. E.g. Wombles can talk. Kitty can't talk. Is Kitty a womble?

We can also try delicately to set up situations in which other logical patterns arise and find out when she does and when she doesn't draw new conclusions.

(Presumably "I'm not a Kitty" wasn't a new discovery at that moment. So she may merely have added it as an interesting observation, not an inference. I think that's part of Vygotsky's theory of development.)

Compare the fallacious reasoning about the racing car and van reported above. The child's 'representational redescription' to support mathematical reasoning about motion and relative speeds was not yet complete.

There may be no normal patterns of development: only individual trajectories through complex terrain, some of them possibly shared with other species that can never tell us (or each other) what they have learnt. So perhaps Ada had reached an unusual 2-year old grasp of at least a subset of logic.

○ **Logic and a jigsaw puzzle**

I once noticed an older child (unfortunately I did not record his age at the time) indicate what appeared to be a kind of understanding of the disjunctive dilemma:

P or Q
not-P
Therefore Q

He was assembling a jigsaw puzzle with help from an adult. Together they had reached the stage where there were two pieces left and two gaps in the puzzle. He picked up one piece and tried fitting it into one of the holes in various orientations, and failed. He then tried the other hole and succeeded. After that, in an exaggerated ceremonious mode he picked up the last piece and as he moved it towards the remaining hole announced "So this piece must go here".

Why did he not say "So this piece goes here" ? Perhaps there was some sort of understanding that the previous success had made something impossible, leaving only one option when there had previously been two. Alternatively, he may simply have noticed the difference between previous situations where each piece could potentially fit into several holes, requiring tests to be done to select the right one (which in some cases can be done perceptually, when a shape is very unusual) and the new situation where there is only one option.

This discussion is merely intended to indicate that we may not have good theories about possible transitions in a child's mind, and therefore are not in a position to use evidence to support one theory.

A note on logic and rules

Logical correctness is often mistakenly regarded (e.g. by philosophers) as conformity with some set of rules. But that cannot be right.

Making logical inferences of the sort we are considering always involves noticing that something is **impossible**. What makes it impossible is not conformity to some rules but structural relationships within the example.

Logicians (starting with Aristotle, or some of his predecessors) notice some kinds of impossibility that other people detect and use unthinkingly (e.g. the impossibility of P or Q true, P false, Q false). So a rule gets formulated: if P or Q is true and P is false, then Q *must* be true. Similar things happen in the discovery of geometrical theorems.

But the rules do not explain the necessity. They merely express discovered generalisations. There are many different philosophical theories about what to say next, including the theory that we create the mathematical truths by adopting the rules. Any working mathematician knows that's false, as did Kant.

(There's much more to be said about this.)

[CONTENTS](#)

○ **From special to general and back again**

Sometimes an individual's advance of knowledge involves noticing that a particular problem is a special case of a general type of problem. Several examples of this are

below.

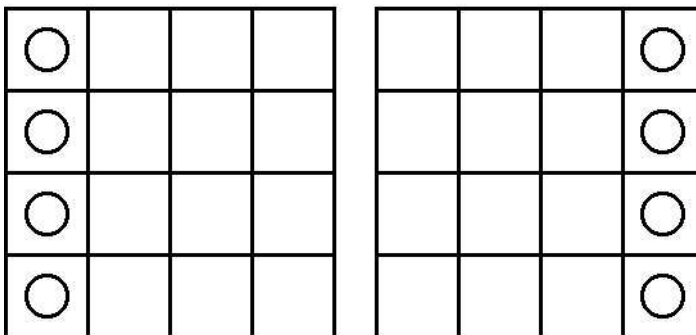
○ **Sliding coins diagonally on a grid**

Below are several puzzles requiring the ability to find possible transformations of a pattern of coins on a grid by sliding the coins diagonally in any direction, i.e. using only these moves:

- Left-Up
- Left-Down
- Right-Up
- Right-Down

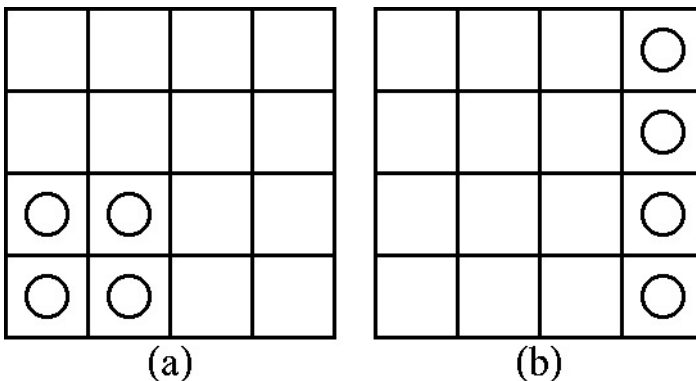
Can you slide the 4 coins from column 1 to column 4 using only diagonal moves?

Using only diagonal slides, can you get
from this to this ?



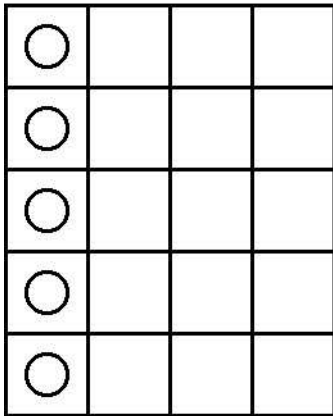
What is the minimum number of moves required?

Does starting from a different configuration change what is possible? Can you get from configuration (a) below to configuration (b), using only diagonal moves?

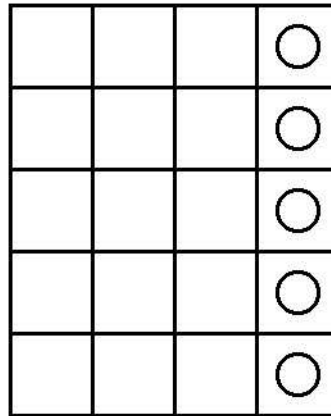


The next one is harder:

Can you slide the 5 coins from column 1 to column 4 using only diagonal moves?
 Using only diagonal slides, can you get
 from this



to this?



What is the minimum number of moves required?

How people work on such problems differs according to prior knowledge and experience.

Sometimes proving that something is impossible can be done by exhaustive search (though understanding the need to ensure that the search is exhaustive is an achievement, as is organising the search so as to ensure exhaustiveness).

A different kind of competence can lead to a much more economical explanation of why the task is impossible.

The core characteristic of mathematical thinking, which frequently motivates new developments in mathematics is **productive laziness**, which I suspect begins to develop between ages 1 and 3 years.

This is a case where the advance of knowledge involves noticing that a particular problem is a special case of a general type of problem.

(If a problem is too hard to solve, trying a harder one sometimes gives new insights.)

If you have not noticed the easy way to solve the above problems consider what difference it would make if the squares were black and white, as on a chess board. Mathematicians can use the notion of "parity" here. E.g. giving squares coordinates, they can be divided into two classes: those whose coordinates sum to an even number and those whose coordinates sum to an odd number. The squares in a horizontal or vertical line will have alternating parity.

Squares in a diagonal line will have the same parity. This makes it very easy to check whether a start configuration can be transformed to a target configuration.

Normally such discoveries are made only by adult or bright mathematical learners. My point is that a young child could learn some of the generative facts about the diagonal moving coin domain by playing. Using a two-colour grid will make some things easier to learn. (Why?)

CONTENTS

- **Pulling an object towards you: blankets, planks and string**

See

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/orthogonal-competences.html#blanket>

- **The surrogate screwdriver example**

Some children when faced with a hard to open flanged lid (e.g. on a large coffee tin) can learn how to use a screwdriver or the back of a spoon to lever the flange up.

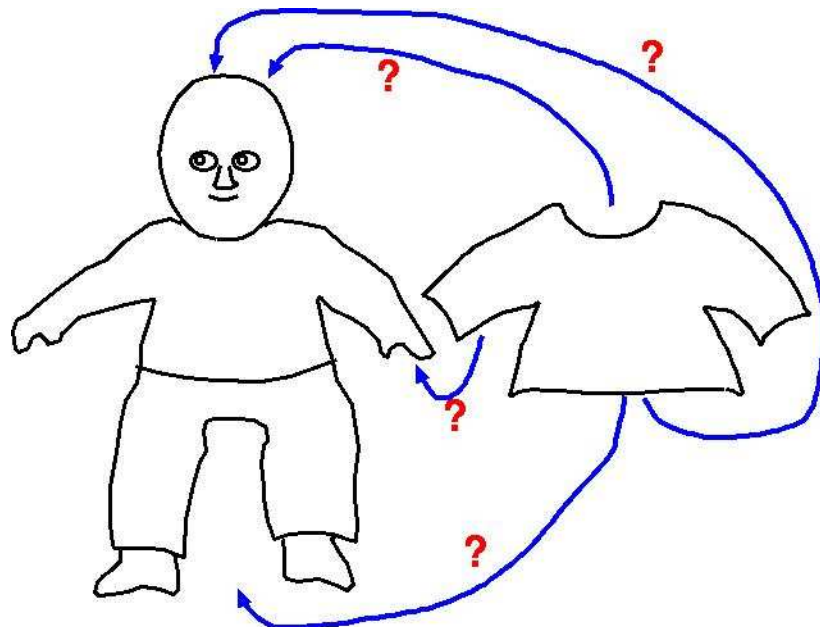
If they cannot find such an object, but they understand what it is about the screwdriver or spoon that makes it a suitable tool, some of them will notice the possibility of using the lid of another tin instead of a screwdriver, to lift the stuck lid.

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/orthogonal-competences.html#lids>

- **Topological and semi-metrical puzzles**

These can form a domain of expertise for older children and adults.

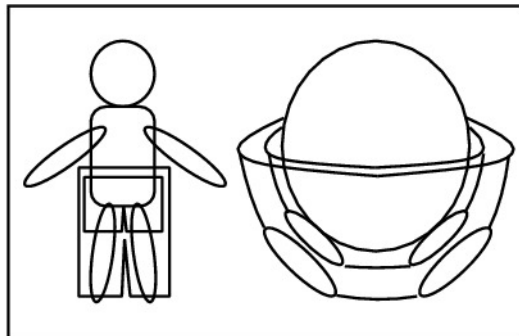
1. **What are good and bad ways to try put a shirt on a child or yourself?**



What sequence of movements could get the shirt onto the child if the shirt is made of material that is flexible but does not stretch much? Why would it be a mistake to start by pulling the cuff over the hand, or pushing the head through the neck-hole? What difference would it make if the material could be stretched arbitrarily without being permanently changed?

The above example is discussed in more detail [here](#).

2. Can Mr Bean remove his underpants without removing his trousers?



The problem of removing underpants without removing trousers, has many variants, but all depend on topological equivalences between different configurations of portions of clothing.

Search for: Mr Bean, Rowan Atkinson, trousers, beach, or watch this video: <http://www.youtube.com/watch?v=ZWCSQm86UB4>

The figure comes from this paper on 'Diagrams in the mind':

<http://www.cs.bham.ac.uk/research/projects/cogaff/00-02.html#58>

3. Impossible transitions involving rings

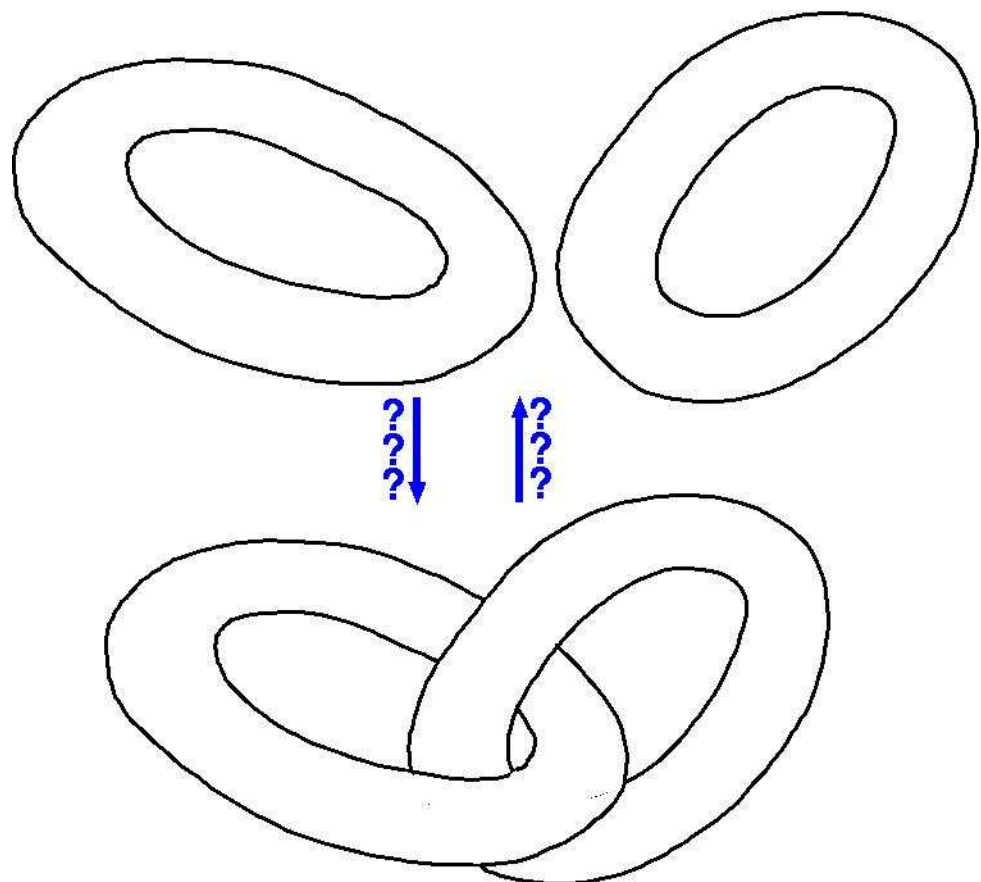


FIG 2 Can two rigid impermeable rings be linked and unlinked?

When children are entertained by a stage magician apparently linking and unlinking rings or loops, this is evidence that they already understand the topological impossibility being demonstrated, even if they lack the vocabulary or the expertise to describe the impossibility or to explain why it is impossible.

I am not aware of any AI system that can be mystified in this way, let alone one that can enjoy being mystified, as children can be. For examples see: <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/rings.html>

4. The "fisherman's folly" puzzle



Starting from the configuration on the left the aim is to get to the configuration on the right, without disconnecting the rope from the two disks at its ends.

(This picture is from the very interesting paper by Cabalar and Santos, below.)

There are many more puzzles shown at the "MrPuzzle" web site

<http://www.mrpuzzle.com.au/>, for example:

<http://www.mrpuzzle.com.au/images/ropes.jpg>

Dealing with such puzzles requires the ability to think about transformations of physical objects that preserve topology, involving flexible inelastic strings, beads, discs, and various rigid objects with holes and slots through which string and other things can pass.

In many cases it is also important to make use of non-topological relationships such as relative size (e.g. a bead is too large to pass through a hole, and a string loop is too short to pass over the far edge of an object).

In such cases, an important kind of discovery is how an alteration that does not transform the topology can transform a metrical relationship. E.g. pulling part of a string from one portion of the puzzle to another portion can increase the size of a loop until some object can pass through it that previously could not.

For each class of puzzle there can be a wide range of possible actions to consider. In particular the learner may need to or discover:

- which describable changes (e.g. causing two rings to become linked, or unlinked, as discussed above) may actually be impossible, and therefore cannot occur as part of a solution to the puzzle,
- which types of action produce which types of effect,
- how to think about what new actions would be made possible by performing

one of the immediately possible actions,

- how to iterate such thinking so as to consider possible sequences of actions,
- how to think about the consequences of a sequence of actions,
- how to remember previously considered and rejected sequences so as not to waste time repeatedly searching down blind alleys.

There seem to be many different domains/microdomains a learner can explore: including the possible processes associated with a particular puzzle, the possible processes associated with a class of puzzles, and the possibilities created by combining features of different puzzles.

For more on such puzzles and formal reasoning about them see

Pedro Cabalar and Paulo E. Santos,
Formalising the Fisherman's Folly puzzle, AIJ, 175,1,pp 346--377, 2011
<http://www.sciencedirect.com/science/article/pii/S0004370210000408>

NB Looking at the sophisticated logical formalism developed in that paper to enable a computer to reason about such puzzles it seems clear that what their AI system does is very different from what a logically and mathematically naive human might do when looking at the same puzzles and thinking about actions that would change relationships, e.g.

"If I push that disk through the slot, I shall then be able to slide the ring up over the top of the post, but..."

Such thoughts seem to make intrinsic use of the structure of the perceived scene in something like the way described in [Sloman 1971](#).

(The 1971 paper made a distinction between "Fregean" representations, where all syntactic complexity represents application of functions to arguments, and "Analogical" representations in which parts of the representation represent parts of what is represented, and properties and relations within the representation represent properties and relations within the thing represented.

It is often assumed that analogical representations must make use of **isomorphisms**, but the paper showed that that is not true. In particular a particular syntactic property or relation (in the representation) can have different semantic functions in different contexts, representing different properties and relations in the scene depicted. That's trivially obvious for 2-D representations of 3-D scenes, since isomorphism is impossible in that case.)

These questions are all related to the question: what sort of understanding of the puzzle (and what form of representation of that understanding) allowed the authors to discover the axioms that characterise it well enough to be used by an AI system? This is also related to the problem of how our ancestors perceived, thought and reasoned about spatial structures and relationships before Euclidean geometry had been codified, and even longer before cartesian coordinates were used to represent geometry arithmetically and algebraically.

It seems very likely that those pre-Euclidean and pre-Logical forms of representation and reasoning are still used, unwittingly, by young children and by other animals with spatial intelligence, e.g. nest-building birds and hunting animals.

- **Placeholder for discussion of knots** A first attempt at posing some mathematical questions about knots, for non-mathematicians is [here](#).
-

- **Learning about numbers (Numerosity, cardinality, order, etc.)**

(An unacknowledged bag of worms.)

It is often thought, or implicitly assumed, that there's a unitary concept of number, such that one either understands what numbers are or does not. Nothing could be further from the truth (as logicians, meta-mathematicians, philosophers of mathematics, and computer scientists have known for some time). I'll try to explain some of the differences and relationships between number domains -- though this is not a complete account.

Numerical competences are widely misunderstood, in part because of a failure to distinguish what could be called "Numerosity" which can be detected as a perceptual feature (related to area, or volume), as opposed to **cardinal number** or **cardinality**, which is inherently concerned with one-to-one mappings (bijections). To a first approximation, this is a difference between recognising a pattern based on two measures (density plus area or volume) and applying a sequential procedure (algorithm) that produces a "result" -- e.g. the result of counting elements of a set. It is also possible in some cases to parallelise (parallelize) that algorithm, e.g. by getting a collection of people all to sit on chairs and then seeing whether any chairs or any people are left over; or checking that two collections of dots are linked by lines where every line joins a dot in one set with a dot in the other, and no dot has more than one line ending on it.

To complicate things, the difference between numerosity and cardinality is much less sharp when the numbers involved are small. (But it is not unusual for different mathematical sequences to have a common "limit".) Another complication is that both numerosity and cardinality are different from, but closely related to various notions of measurement along a scale, used in science and engineering such as the "ordinal", "interval" and "ratio" scales distinguished by S.S. Stevens in 1946 as explained in http://en.wikipedia.org/wiki/Level_of_measurement Unfortunately, he also applied a notion of 'scale', which he called a "nominal scale", to an inherently un-ordered collection of labels.

Another distinction that can be made among scales is between orderings (using relations "more", "less" and "same") that are discrete, as in sizes of families, and those allow continuous variation, as in length, area, volume, mass, etc. The orderings need not be total since some cases may be incomparable, in which case a "partial" ordering exists. It is possible to define these concepts with great precision, but for people who are unfamiliar with the required formal concepts it is easy to confuse the different sets of relationships, or worse, to assume that there is one concept of number which an individual does or does not have.

One of the features of toddlerhood is that the early stages of all of these important and importantly different systems of concepts developed without the learners or their parents or teachers having any idea that such a complex set of structures is being constructed.

And that's before there's any learning about negative numbers, fractions or a mathematically precise notion of the real continuum.

At present I don't think we have an adequate collection of information-processing models to represent the different processes of construction in different domains (e.g. tactile, auditory, visual, and motor control domains) and the powerful mechanisms of abstraction that unify them into different families, so that, for example, 'more' and 'less' can be applied to height, width, angle, area, spatial volume, rotation, linear or rotational speed, weight, force, acoustic properties (e.g. pitch and volume), motor properties (pushing, pulling, twisting, or bending more or less hard, etc.), and grasping the differences between processes where becoming more or less are continuous processes and those where they must be discrete, both of which allow reduction to a "zero" case (an empty set, an infinitely small length or angle or speed) or the opposite extreme (getting "more and more" X indefinitely).

Most empirical or modelling research latches onto some small subset of relationships in this rich and tangled (but ordered!) network without the researchers understanding what they are not attending to.

For now I'll end with a few comments on two sets of concepts that are particularly often confused, or if the difference is noticed it is not described accurately.

- **Numerosity and cardinality**

There is a notion of numerosity that can be thought of roughly as an estimate of product of area (or volume or temporal extent) and spatial (or temporal) density. Two groups of dots will have different numerosity if the area is the same but the dots are more dense in one group, than in the other, or if the density is the same but one group is larger than the other. Likewise two sequences of beeping noises can be compared as to their frequency (spatial density) and their temporal duration. The product of density and size or duration gives an estimate, but in general **only** an estimate of the cardinality of the set of objects or events.

When the set of items exists in the environment that estimate can be right or wrong: there will a definite number of them.

But when the items are experiences, e.g. experienced sounds, or texture elements, the sophistication of the perceptual processing mechanisms in producing these experiences may not allow there to be a definite number of elements. For example, even if there is a definite number of stars and planets visible in the sky from a particular location, it does not follow that a human or other animal looking at that sky has a definite number of starry experiences. (This is one of several reasons why an information-processing account of "qualia" requires a kind of detail that's missing from every theory of mind I have ever encountered.)

One problem with a concept of numerosity based on combining (a) an ability to detect and estimate density and (b) an ability to detect and estimate some sort of spatial or temporal extent (of a linear interval, an area, a volume, a temporal interval, etc.) is

that when the density varies across the items, then an average density has to be computed to get a measure for the whole set. Since density is already an average, that requires averaging a spatially varying average -- a non-trivial computation. Another problem is detecting whether two densities or two areas are the same. The larger the areas the harder it may be to compare densities accurately. In particular, the harder it is to tell if A's numerosity is greater than B's. So more dots may need to be added to a large collection to make the size difference noticeable. This means that the graph of perceived numerosity against actual cardinality flattens out as cardinality increases. This may take a logarithmic form.

(I have no idea whether anyone has actually investigated which of these computations brains are capable of, for which modes of sensory input.)

If a child (or animal) with an ability to estimate numerosity as described above, perceives two groups G1 and G2 which have both different sizes and different densities comparing numerosity is much harder than where G1 and G2 have the same density, or the same extent. If the density is roughly uniform within each group, and if the perceiver can compute numerical values for both density and area or volume, then the two numbers can be multiplied to provide an estimate of numerosity. The ability to multiply seems to require a prior grasp of numbers, but that can be avoided if the multiplication is done by dedicated, domain specific machinery. In that case, there can be no comparison of numerosity of a sequence of heard sounds and numerosity of dots scattered around an area.

However when both numbers are small they can be compared directly by some form of counting, or setting up a one to one correspondence between the sounds and the dots. That will show if there are more of one than the other. So in that case the ability to estimate cardinality directly removes the need to compute numerosity by performing a multiplication of density and extent.

It seems that humans can compute and compare numerosities from quite an early age (e.g. before being able to count), but they get better as they grow older (and presumably have more experiences of numerosity judgements), and also gradually get a better meta-cognitive understanding of what they are doing. Before that, as Piaget showed, they can display extraordinary confusions because they don't yet have a concept of cardinality as something that is conserved as objects are packed closer together or spread out more.

If the distribution of items in the space is highly irregular the task of comparing numerosities can become very difficult, and in some cases deceptive. There's a lot more to be said about numerosity, but for now the main point is that it is a totally different concept from cardinality, which is fundamentally connected with the notion of a one to one mapping, and researchers who don't make this distinction often write as if there were just one concept of number.

- **Understanding cardinality** There are many ways in which the domain of cardinality can be approached. One route, making important use of a learnt sequence of sounds ("one" "two" "three" ...), later followed by a systematic method of generating additional members of the sequence, is often followed by children in our culture. Some of the processing requirements for such competences are described in chapter 8 of *The Computer Revolution in Philosophy* (1978) <http://www.cs.bham.ac.uk/research/projects/cogaff/crp/chap8.html> independently identified by Heike Wiese (Potsdam University) in her 1997 PhD Thesis. <http://www.uni-potsdam.de/u/germanistik/fachgebiete/geg-spr/page.php?id=hwiese&spr=1>

The following seems to be a fairly standard (but mostly unnoticed by researchers??) way of acquiring cardinality competences, though these components are not learnt in sequence, but interleaved:

- Learn some fixed sequence of verbal actions, e.g. saying "one, two, three, ..."
- Learn various ways of extending that sequence e.g. "ten, eleven, twelve, ... twentyone, twentytwo, ... a hundred and one, a hundred and two, etc." (Later replaced by more compact written forms.)
- Learn to perform other discrete sequences of actions, possibly related to discrete collections of items in the environment, e.g. tapping or pointing at items on a table, touching successive rails in a railing, stepping onto successive steps on a staircase, moving successive objects from one container to another, making successive marks on paper or on a blackboard, ...
- Learn to say the numeral sequence in parallel with performing another discrete action sequence.
- Learn to do them in synchrony (sometimes it is easier to start with synchrony).
- Learn to detect and repair failures of synchrony, omissions, repetitions of either the numerals used, or the other actions.
- Learn to use different stopping conditions for the synchronous production, e.g. so as to answer "How many Xs are there?" "Are there more Xs than Ys?" "Do we have enough Xs for everybody?" "Please move N Xs into (or out of) the box", "Please get one X (e.g. plate) for each Y (e.g. person)"
- Learn to treat a set of numbers or numerals as Xs or Ys, e.g. "How many numbers are there between 7 and 12?"

A child with those competences organised into a deductive system has the basis for making an infinite collection of new discoveries.

E.g. If counting Xs produces the number 5, what will happen if they are counted in the opposite direction? At a certain stage the child will not know, without trying. The answer is discovered empirically.

At a later stage the child will think that the question is stupid. What exactly is that transition? Does anyone have any idea what changes in brains, are required to produce that insight?

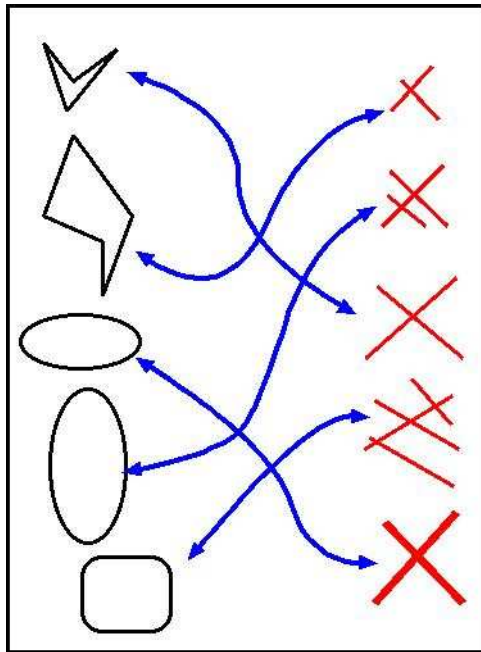
○ **Evolutionary origins**

It is only recently that mathematicians and logicians have developed explicit ways of talking about and reasoning the various structures, relationships and processes mentioned above. Everything that is now taught explicitly in mathematics classes or informally in games and social activities must at some point have been learnt by individuals who had no teachers that had already made these discoveries. So biological evolution must have produced the precursors, not forms of teaching or social interaction.

I conjecture that much of what happened in our ancestors to enable them to make these discoveries is still going on unnoticed in young children (and some other animals) as they play with and gain various kinds of mastery over, their environments. In that case, by the time we start teaching mathematics to children in school we are

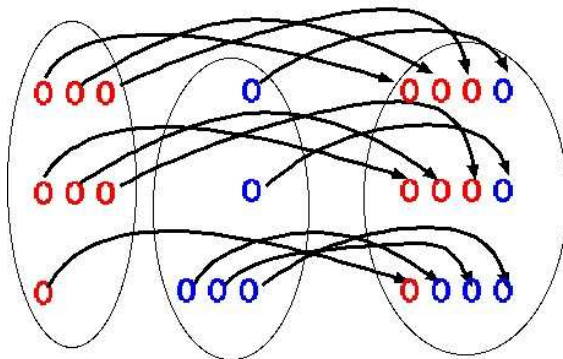
using sophisticated apparatus about which teachers know nothing, or very little. So they have no idea why or how their teaching works. Neither do developmental psychologists.

- **Learning about one to one mappings (bijections)** What cognitive developments are required for a child to be able to make use of one-one mappings and reason about them? E.g. if everyone is seated at a chair round a table, and there are no spare chairs, then if the people walk around and then sit at any chairs everyone will be seated with no spare chairs?



If all the strings connecting objects on the left have their ends swapped, the same objects will still be connected by the same strings.

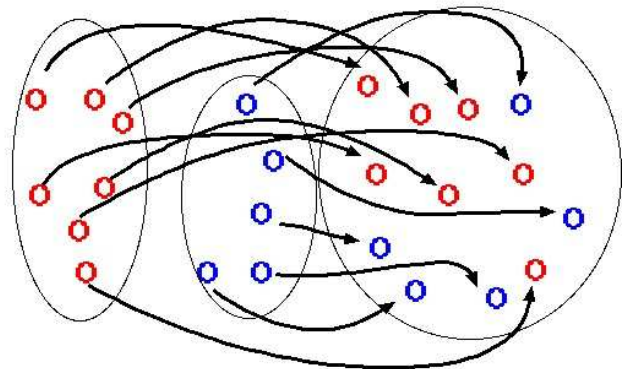
If the connections on the left ends of strings are preserved, but the right ends are detached and rearranged, how many different ways are there of connecting the ends on the right to objects on the right?



Then rearrange the items, leaving the strings attached.

Is it 'obvious' that the correspondence defined by the strings will be preserved even if the strings get tangled by the rearrangement?

Join up corresponding items with imaginary strings.



○ **Steps and slides towards infinity (Added 19 Mar 2014)**

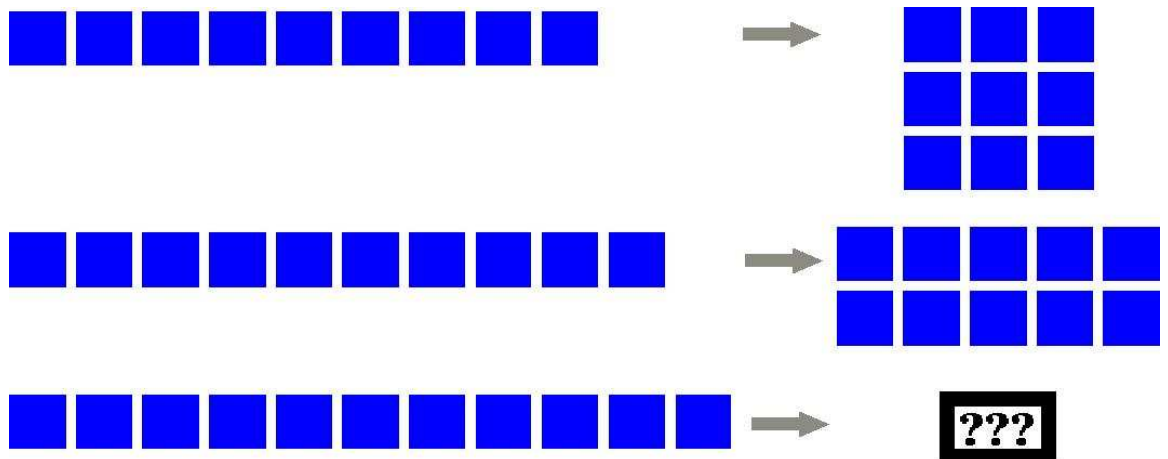
This section has been moved into a separate file, where it will be (gradually) extended (but not indefinitely!).

○ **Rearranging blocks and discovering primeness**

A child given a set of wooden cube-shaped blocks can do all sorts of experiments -- exploring the space of processes involving the blocks.

- Some of the experiments involve learning about the material of which the blocks are made.
- Some involve learning about types of physical interactions -- e.g. what happens when you bang two blocks together, or throw a block, or push one over the edge of the table, or what difference it makes whether the floor has a carpet or not when you are trying to build towers, or what happens if you put a block in a cup and shake it in various ways.
- Some of the experiments may lead to discoveries of properties of **arrangements** of various kinds. E.g. if a group of blocks is separated from the rest the elements of the group can always be arranged in a line (if there's space on the table, or on the floor, or in the room,...). But sometimes the blocks can be arranged into other configurations, e.g. a square frame, a rectangular frame, an rectangular array.

Then the child may notice that attempts to rearrange some configurations, e.g. a configuration of 11 blocks, into a rectangular arrangement always fail: What kind of experimentation can that provoke, and what sorts of discoveries can be made?



How could one be sure that there is NO way of arranging the last collection into a rectangular array, apart from the straight line shown?

Could such a child discover the concept of a prime number?

When I discussed this hypothetical example (discovering theorems about factorisation and prime numbers by playing with blocks) with some people at a conference, one of them told me he had once encountered a conference receptionist who liked to keep all the unclaimed name cards in a rectangular array. However she had discovered that sometimes she could not do it, which

she found frustrating. She had unwittingly discovered empirically that some numbers are prime, though apparently she had not worked out any mathematical implications.

Could the child rearranging blocks discover and articulate the fundamental theorem of arithmetic? (The unique factorization theorem.)

Are some forms of mathematical discovery impossible without a social environment?

Don't assume a teacher with prior knowledge of the theorems has to be involved: someone must have made some of these discoveries without being told them by a teacher.

NOTE 1 One of the fundamental requirements for mathematical thinking is being able to organise collections of possibilities and making sure that you have checked them all.

If you can't do that you don't have a mathematical result, only a guess.

NOTE 2 (9 Aug 2012) I have just discovered that this kind of discovery of primeness by a computer program was discussed in Alison Pease, Simon Colton, Ramin Ramezani, Alan Smaill and Markus Guhe, Using Analogical Representations for Mathematical Concept Formation, in Eds. L. Magnani et al, *Model-Based Reasoning in Science & Technology*, Springer-Verlag, pp. 301-314, 2010, http://homepages.inf.ed.ac.uk/apease/papers/pease_mbr09.pdf

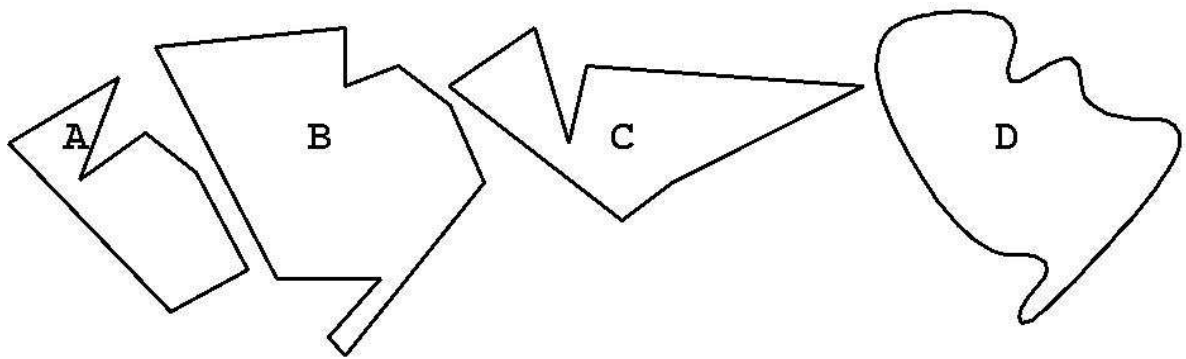
○ **Learning about measures**

Going from the cardinality competences requiring use of one-to-one mappings to understanding that physical things in the environment can have **measures** (e.g. length of a straight line, length of a curved line, area of a rectangular shape, area of a curved convex shape, area of a curved shape with concavities, area of a shape with holes, volume of a rectangular block, volume of a curved 3-D shape, etc.) involves a collection of major transitions which required major advances in the history of mathematics -- including the invention of integral and differential calculus.

It is not always noticed that without the sophisticated apparatus of modern mathematics many measures form only **partial** orderings.

E.g. at a certain stage areas or volumes may be comparable only if one shape can fit entirely inside another. So a long thin rectangle and a circle whose radius is less than the length and greater than the breadth of the rectangle are not comparable in area, at that stage. (As far as I know this was ignored by Piaget and all the researchers inspired by his work.)

For example, several different competences are required in order to rank the areas **A**, **B**, **C** and **D** in the following figure.



Someone who can accurately visualise the effect of moving one bounded area while another remains fixed, or who can cut out the area and move it onto another, may discover that area A can fit entirely inside B. So the area of A is less than the area of B.

However, the shape A cannot be contained in C, and C cannot be contained in A. Moreover, C cannot be contained in B, and B cannot be contained in C. This means it is impossible to rank shapes A, B and C in area on that criterion. They form only a partial ordering relative to the containment criterion.

Someone who has solved the non-trivial problem of assigning measures of area to rectangular shapes, and then discovered that that can be extended to a way of assigning measures to triangles:

$$\text{area} = \text{half}(\text{base} \times \text{height})$$

might then discover (how?) that any area bounded by straight edges (i.e. any polygon) can be systematically divided into triangles, so that the area can be computed by triangulation, followed by summing all the areas of the triangles. That will enable each of the three shapes A, B, and C to be given a numerical measure of area, instead of just a partial ordering of spatial extent defined in terms of containment.

But a polygon can be divided into triangles in different ways, so the argument assumes that different triangulations of the same total area will produce triangles whose sums are all the same. Is that obviously true? (It may seem to be obvious if you start from the assumption that the measure of area of an arbitrary shape is uniquely defined. But that assumption requires justification. In fact there is a lot of non-trivial mathematics concerned with investigation of things that seem obvious to non-mathematicians.)

If we attempt to generalise the notion of area to a region not bounded by straight lines, like figure D, then there is no way to convert that region into a set of triangles. Our simple partially ordered notion of relative area defined by containment can still be used. For example, figure A can be re-located to fit entirely inside D, though that may not be obvious to everyone.

If, however, we wish to extend the notion of a **measure** of area, so as to provide a **total** ordering of areas that includes shapes with curved boundaries, like D, then a different approach is required. In fact it requires the use of integral calculus and concepts of limits of infinite series, which were invented by geniuses like Newton and

Leibniz and not fully clarified until the mathematics of the 19th Century. (Some might say: not even then!).

There are also problems about the justification for talking about cardinality of large collections of objects (like the visible stars on a clear night, or the leaves on a big tree) where we do not have any chance of counting them, e.g. because they exist for a very short time, or because they are in constant motion, or for some other reason.

All this means that when researchers ask whether children or animals have concepts of size or number they often have no idea of the variety of interpretations that their question can have, with different answers being appropriate to the different interpretations. It is probably fair to say that most members of the adult population of any country on this planet lack well-defined concepts of area and volume. (It may be assumed that area and volume can always be defined in terms of the results of weighing, but that typically assumes the notion of uniform density, which in turn assumes notions of weight and volume.)

It is not clear which of these competences (relating to cardinality, mappings and measures) a child can acquire without help. The ontologies required, the invariants, and the applications, all must have been discovered originally piecemeal, perhaps in inconsistent fragments, without help, and then organised into a shared system through some collaborative process, probably over many generations, long before Euclid's time. I don't know if we'll ever find definitive evidence for those aspects of our pre-history. But perhaps we can replicate some of them in future intelligent robots. And if we look carefully, asking the right questions, we may be able to see some of the fragments in child development, though not all fragments will necessarily appear in all children: there are many routes through this maze of ontologies.

There is further discussion on related topics in this 2010 workshop paper:

<http://www.cs.bham.ac.uk/research/projects/cogaff/10.html#1001>

If Learning Maths Requires a Teacher, Where did the First Teachers Come From?
Symposium on Mathematical Practice and Cognition 29th - 30th March, 2010, De Montfort University, Leicester, AISB 2010 Convention

- **27 Aug 2012: Need to add 'modulo numbers'**
(Reminded by Alan Bundy, 27 Aug 2012)

Alan Bundy has reminded me that some children learn from clock faces and other structures that it is possible to do a kind of counting that goes up to a certain number and then re-starts from 1, for instance reciting the numbers on an old-fashioned clock face.

For mathematicians, this is a special case of 'modulo' arithmetic, namely arithmetic in which there is only a finite set of numbers and counting beyond the largest number always starts again from 1.

For example, $3+4$ modulo 5 is 2, $3+4$ modulo 6 is 1, $3+9$ modulo 6 is 0.

If we assign numerical coordinates to rows and columns of a chess board, then associate each square on the board with the sum of its coordinates, then the bottom left 3x3 corner would have these numbers:

456
345
234

However, if each square is associated with the number

sum of coordinates modulo 2

then the bottom left corner would have a different collection of associated numbers with new symmetries:

010
101
010

● The chocolate slab puzzle

This is an old and well known puzzle to which new "wrinkles" are added below.

○ Statement of the puzzle

You have a slab of chocolate in the form of a 7 by 7 square of pieces divided by grooves, and you want to give 49 friends, each one piece. You have a knife that can cut along a groove. What is the minimum number of groove cuts that will divide the bar into 49 pieces? RULES FOR CHOCOLATE CHOPPING: Stacking or overlaying two or more pieces, or abutting two pieces, to divide them both with one cut is not allowed: each cut is applied to exactly one of the pieces of chocolate.

The puzzle draws attention to a domain of processes of subdivision of a rectangular array into its component elements by a succession of linear slices.

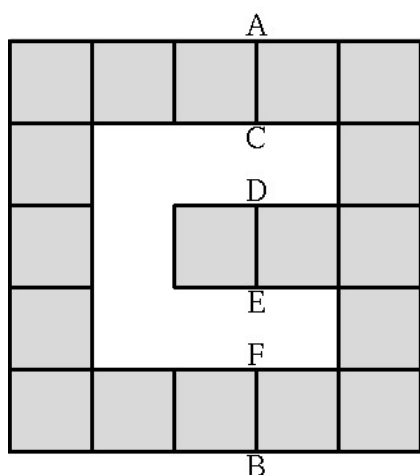
- You can play around with lots and lots of 7x7 bars trying different methods.
NOTE: the number of possible ways of doing the division is pretty big!
- You can also do it for different sized slabs, e.g. 2x5, 6x3, 4x4, etc.
- You may start noticing a pattern for each size: E.g. a 2x5 array always requires 9 cuts, a 6x3 array always requires 17 cuts, etc.
- If you run out of chocolate slabs you can test the pattern out on more examples, using squared paper instead of chocolate slabs.
- You may realise that the number of cuts does not depend on what the material is that you use or what the knife is made of: only "the structure" of the process (an abstract pattern), not the material operated on, matters. (Why?)
- You may also notice that in addition to the pattern for each size there is a more general pattern that applies to all the sizes. Namely: each cut produces one more piece. So it makes no difference in which order you make the cuts or where you make them, the number is always the same.

A learner can eventually see WHY the result generalises for all possible rectangular blocks, though that requires a type of information processing architecture that, as far as I know, no current AI robot has.

So they can make empirical discoveries but cannot make mathematical discoveries.

○ **Discovering counter-examples to the chocolate slab theorem**

It is often wrongly assumed that the necessity of mathematical results implies or requires **infallibility** of mathematicians. That ignores the richness of domains with mathematical properties and the possibility of failing to notice some of that richness. For example a child might discover that a slab like this could be an exception to the theorem.



It's an exception because the original argument assumed that **every** cut divides one piece into two pieces.

With holes, is it a slab or isn't it?

- If you call it a slab, and allow that a cut from A to B is consistent with the rules, then the key "pattern" that each cut produces one more piece does not apply here: there are two new pieces.
- If you say that a cut must join two boundary points and not cross a boundary, then consider the cut from A to C, or from F to B: that leaves the same number of pieces as before
- The "cut" joining boundary points C and D has no effect at all: only the DE cut produces one new piece.
- You can eliminate cases like this by stipulating that the initial slab must not contain any holes.
- However an unusual (non-convex) outer boundary for the slab can produce yet more counter-examples: finding them is left as an exercise for readers.

Often a proof in mathematics that seemed valid works for a range of cases, but has counter-examples not thought of when the proof was constructed, or when it was checked.

Many such examples connected with the history of Euler's theorem about plane polyhedra were discussed in this famous book.

Imre Lakatos: Proofs and refutations: The Logic of Mathematical Discovery
Cambridge University Press, 1976

One of the consequences of our ability to perceive, imagine, or create instances of novel possible configurations is that we can sometimes create new configurations that refute our mathematical conjectures, generalisations or even proofs.

This is different from the empirical refutation of "All swans are white", which turned up in Australia.

○ **Another "Lakatosian" counter example to the chocolate theorem**

When I presented some of these ideas at Liverpool University on 21 Jan 2008, Mary Leng pointed out a possibility I had not noticed: making **partial** cuts.

In defining the problem, I had not noticed the need to specify that every cut must go from one boundary point to another: i.e. no cuts may begin or end at a point that is completely surrounded by chocolate.

This example illustrates the relationship between (a) simple everyday activities, and variations that are clearly intelligible to ordinary people with no knowledge of abstruse mathematics, and (b) deep concepts from topology.

Alison Sloman later pointed out that the counter-example might have been ruled out in advance by requiring portions of the slab to be **broken** rather than **cut**.

It is important not to inflate Lakatos' argument in *Proofs and Refutations* as demonstrating that there is never any real progress in mathematics, or that mathematics is empirical.

On the contrary, every mistake that leads to a revision of a definition, or a statement of a theorem, or a proof adds to our mathematical knowledge: mathematicians can make non-empirical discoveries without being infallible.

○ **Imre Lakatos**

The importance of Imre Lakatos' writings on Science and mathematics.

- Proofs and Refutations
- Falsification and the methodology of scientific research programmes

The most important philosophical point arising out of his survey of the history of Euler's theorem about polyhedra is, arguably, that just because mathematical knowledge is about necessary truths, not contingent truths, is not empirical, and is also not trivial (analytically provable on the basis of definitions plus logic and nothing else), it does not follow that mathematical discovery processes are infallible. On the contrary, mathematicians can make mistakes, and can often discover that they have made mistakes, and patch them.

The same is true of toddlers who (unwittingly) discover and use theorems.

● **Crank Handle and Credit Card**

At a conference in 2011 Alex Stoytchev showed me two videos. One was of a robot that had been trained so that it could use its hand to smoothly turn a crank handle, rotating the axle connected to it (though the robot probably did not know anything about the axle).

The other video was of a toddler standing near the left edge of a closed door holding a credit card so that it was in the vertical slot between the door and its frame. He smoothly moved the card up and down in the slot. Then, apparently unprompted, he noticed the slot on the opposite edge of the door and inserted it there and moved it up and down smoothly. The first configuration required his arm to move up and down roughly in front of him. Because he did not move across to the opposite edge, the second action involved his right arm being extended away to the right, producing a very different geometric configuration and pattern of changes of joint angles and forces required to move the card vertically.

He did not seem to need to learn how to produce the new motion. My guess is that he was not controlling the card by aiming to modify joint angles or aiming to produce specific sensory motor signal patterns. Rather in each case he knew in which direction (in 3-D space) the card had to move, and because it was constrained by the slot it was in, all he had to do was apply a force roughly upward or downward using a compliant grip that allowed the sides of the slot to provide the required precision (a toddler theorem). Applying a vertical force requires different motor signals in different arm positions but visual and haptic/proprioceptive feedback would suffice to control the motion.

I asked Alex what would happen to the robot if it were moved some way to one side, so that turning the crank required a new collection of angles, forces, etc. He said it would fail and would have to be retrained.

I presume that's because the robot had not worked out the toddler theorem that to move a crank handle you need to keep adjusting the force so that it is in the plane of rotation but perpendicular to the line from the axle to the handle. Instead, all it had learnt was statistical correlations in its sensory-motor signals. It was stuck with a **somatic** ontology, whereas it needed an **exosomatic** ontology, in order to exercise off-line intelligence, as discussed [above](#).

The little boy almost certainly used an exosomatic ontology both in formulating his goals and in controlling his actions. Why did he want to perform those actions? I expect that was an example of architecture-based, not reward-based, motivation, described in Sloman (2009)

Note added 21 Apr 2015

Alex and colleagues have a paper on a robot learning to slide a credit card in a vertical gap:

http://home.engineering.iastate.edu/~alexs/papers/ICRA_2012/ICRA_2012.pdf

Learning to Slide a Magnetic Card Through a Card Reader

Vladimir Sukhoy, Veselin Georgiev, Todd Wegter, Ramy Sweidan, and Alexander Stoytchev

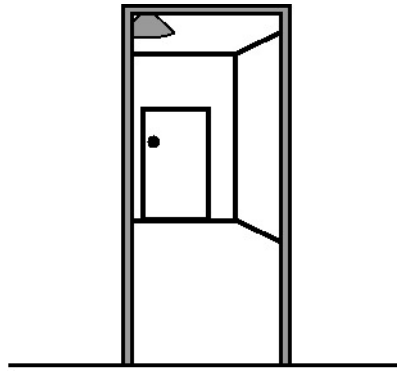
Presented at ICRA 2012, with an associated video:

http://home.engineering.iastate.edu/~alexs/papers/ICRA_2012/ICRA_2012_video.mp4

Here is a more recent video reporting on work in his lab.

<http://www.sciencechannel.com/tv-shows/brink/videos/brink-robots-become-human/>

- **Learning about epistemic affordances**
Getting information about the world from the world, and making the directly available information change.



Things you probably know, but did not always know:

- You can get more information about the contents of a room from outside an open doorway
 - (a) if you move closer to the doorway,
 - (b) if you keep your distance but move sideways.
 Why do those procedures work? How do they differ?
- Why do perceived aspect-ratios of visible objects change as you change your viewpoint?

A circle becomes an ellipse, with changing ratio of lengths of major/minor axes.
Rectangles become parallelograms
- In order to shut a door, why do you sometimes need to push it, sometimes to pull it?
- Why do you need a handle to pull the door shut, but not to push it shut?
- Why do you see different parts of an object as you move round it?
- Why can you use your experience moving round a house to predict your experiences when you move round it in the opposite direction? (Example due to Immanuel Kant).
- When can you avoid bumping into the **left** doorpost while going through a doorway by aiming further to the **right** -- and what problem does that raise?
- How you could use the lid of one coffee tin to open the lid of another which you cannot prise out using your fingers? (Also mentioned above)

● Aspect graphs and generalised aspect graphs

The idea of an "aspect graph" can be viewed as a special case of a domain of actions related to changing epistemic affordances (as defined above).

That's not normally how aspect graphs are presented. Normally the aspect graph of an object is thought of as a graph of topologically distinct views of the object linked by minimal transitions. For example as you move round a cube some changes in appearance will merely be continuous changes in apparent angles and apparent lengths of edges, but there will be discontinuities when one or more edges, vertices or faces goes in or out of view. In the aspect graph all the topologically equivalent views are treated as one node, linked to neighbouring nodes according to which movements produce new views, e.g. move up, move down, move left, move diagonally up to the right, etc. For a non-convex object, e.g. an L-shaped polyhedron the aspect graph will be much more complex than for a cube, as some parts may be visible from some viewpoints that are not connected by visible portions.

Here's a useful introduction By Barb Cutler:

http://people.csail.mit.edu/bmcutler/6.838/project/aspect_graph.html

Some vision researchers have considered using aspect graphs for recognition purposes: a suitably trained robot could see how views of an object change as it moves, and in some cases use that to identify the relevant aspect graph, and the object. (Related ideas, without using the label "aspect graph" were used by Roberts, Guzman and Grape for perception of polyhedral scenes in the 1960s and early 1970s, though the scenes perceived were static.)

However, for complex objects aspect graphs can explode, and in any case, we are not concerned with vision but with understanding perceived structures. A perceiver with the right kind of understanding should be able to derive the aspect graph, or fragments of it, from knowledge of its shape, and use that to decide which way to move to get information about occluded surfaces.

In 1973 Minsky introduced a similar idea for which he used the label "Frame system".

<http://web.media.mit.edu/~minsky/papers/Frames/frames.html>

A few years ago, in discussion of plans for the EU CoSy project

<http://www.cognitivesystems.org/>, Jeremy Wyatt suggested an important generalisation.

Instead of considering only the effects of movements of the viewer on changing views of an object we could enhance our knowledge of particular shapes with information about how things would change if other actions were performed, e.g. if an object resting on a horizontal surface is touched in a particular place and a particular force applied, then the object may rotate or slide or both, or if there is a vertical surface resisting movement it may do neither.

This suggested a way of representing knowledge about the structure of an object and its relationships to other surfaces in its immediate environment, in terms of how the appearance of the object would change if various forces were applied in various directions at various points on the surface, including rotational forces.

This large set of possibilities for perceived change, grouped according to how the change was produced, we labelled a "Generalised Aspect graph". This would be even more explosive than the aspect graph as more complex objects are investigated. For various reasons, we were not able to pursue that idea in the CoSy project (though a subset of it re-emerged in connection with learning about the motion of a simply polyflap in work done by Marek Kopicki).

In currently favoured AI approaches to perception and action the standard approach to use of generalised aspect graphs would require a robot to be taught about them in some very laborious training process.

In the context of an investigation of "toddler theorems" the problem is altered: how can we give a robot the ability to understand spatial structures and the effects of forces on them so that instead of having to learn aspect graphs, or generalised aspect graphs, it can **derive** them, or fragments of them, on demand, as part of its understanding of affordances.

That, after all, is what a designer of novel objects to serve some purpose needs to be able to do.

However, in order to reduce the combinatorics of such a derivation process I suggest that the representation of objects used to work out how they would move, should not be in terms of sensory-motor patterns (not even multi-modal sensory-motor patterns including haptic feedback and vision), but in terms of exosomatic concepts referring to 3-D structures in the environment and their surfaces and relationships, independently of how they are perceived.

Prediction of how a perceived scene would change if an action were applied would take two major steps: first of all deriving the change in the environment, and secondly deriving the effect of that change on the visual and tactile experiences of the perceiver. Among other things that would allow reasoning to be done about objects that are moved using other held objects, e.g. rakes, hammers, and also reasoning to be done about what other perceivers might experience: a necessary condition for empathy.

This is a complex and difficult topic requiring more discussion, but I think the implications for much current AI are deep, and highly critical, since so much work on perceiving and producing behaviour in the environment does not yield the kind of **understanding** provided by toddler theorems, an understanding that, later on, can grow into mathematical competence, when generalised and articulated.

CONTENTS

- **Playing and exploration can be done in your mind instead of in the world** It is often forgotten that perceptual apparatus can provide information not just about what exists in the environment, but also about what is possible in the environment.

Having discovered those possibilities an animal, or robot, can play with them, e.g. by trying various combinations of possibilities to find out what happens.

We can play in the environment, and we can play in our minds.

- **Playing can reveal both new possibilities and impossibilities.**
(Discovering constraints.)

Both kinds of experimentation can increase know-how, and support faster problem-solving, using patterns that have been learnt and stored. But we need to account for the differences between learning that is empirical and learning that is more like deductive reasoning, or theorem-proving. (As in "toddler theorems" about opening and shutting drawers and doors, or pulling a piece of string attached to something at the other end.)

- **Hoop theorems (Added 9 Aug 2012)**

I noticed a very young child (age unknown, though he could stand, walk, and manipulate a hoop, but looked too young to be talking) playing with a hoop on a trampoline.

He seemed to have learned a number of things about hoops, including

- If you hold a hoop horizontally in front of you with two hands, then move your hands in a certain way the hoop will go over your head: you can then release the hoop and it will fall to the ground, eventually encircling your feet.
 - If you want to make a hoop roll you can hold it in a vertical plane perpendicular to your stomach, with a part resting on the ground in front of you and one hand on each side near the top of the hoop, then move both hands rapidly away from you and then apart. This discovery is a subtle mixture of empirical and non-empirical discoveries. For example, the strategy depends on empirical facts about the material of which the hoop is made, including its approximate rigidity and a uniform distribution of mass around it, which the child cannot be expected to have understood.
-

- **Carrying things on a tray**

Why it is easier to carry a tray full of cups and saucers using a hand at each side than using both hands on the same side?

Why is it easier with two grasp points than with only one?

- **Domains and micro-domains concerned with meccano modelling**

See: <http://www.cs.bham.ac.uk/research/projects/cosy/photos/crane/> To be expanded later.

- **POLYFLAPS - An artificial domain for research in this area**

Many of the domains in which a child or animal learns are products of biology, physics, chemistry, the weather, etc. But others are products of cultures, e.g. domains related to clothing, eating utensils, toys, games, etc.

For the purposes of research in intelligent robots, we have created an artificial domain in which humans may have as much to learn as the robots, and which can start simple, then get increasingly complex: the domain of polyflaps. See <http://www.cs.bham.ac.uk/research/projects/cogaff/misc/polyflaps>

- **Doorframe-climbing theorems**

These two videos show a three-year old girl, Sofya, climbing up a door-frame using friction between hands and frame and between feet and frame.

<http://www.youtube.com/watch?v=cij-cT5zkHo>
Early, partially successful attempts.

<http://www.youtube.com/watch?v=FmH8jFLrwDU>
Fairly expert performance.

It is very unlikely that Sofya has had to learn every possible combination of sensory inputs and motor outputs required to ascend the door-frame. Rather she has (almost certainly) grasped a number of general principles common to classes of states that can arise, using an exo-somatic ontology (i.e. referring not to what's going on inside her skin, but which surfaces are in contact with which and how the contact varies).

She never tries moving both feet up at the same time -- instead always ensuring that two hands and one foot are applying enough pressure to hold her up while she moves the other foot to a higher location. Has she discovered a toddler-theorem about how stable (or nearly stable) configurations differ from unstable ones?

There are also subtle ways in which she adjusts the pressures in order to start sliding down, as opposed to falling down.

It seems that this performance makes use of some learnt generalisations about how things should feel and some more abstract inferences about how things should be configured.

- **Window (or door) opening theorems** Why do you often have to move a handle down or up before you can push a window open?

What about doors?

Added 7 Aug 2013: ROBERT LAWLER'S VIDEO ARCHIVE

Bob Lawler has generously made available a large collection of video recordings of three children over many years here: <http://nlcsa.net/>

I have not yet had time to explore the videos in any detail, but I expect there are many examples relevant to the processes and mechanisms involved in discovery of toddler theorems.

The first video I selected at random

<http://nlcsa.net/lcla-nls/lcla-video/> "Under Arrest"

illustrated many different things simultaneously, including how two part-built information processing architectures at very different stages of construction, with an adult out of sight, could interact in very rich ways with each other, some physical some social, and to a lesser extent with the adult through verbal communication. The older child clearly has both a much richer repertoire of spatial actions and a much richer understanding of the consequences of those actions. He also has some understanding of the information processing of the other child, including being able to work out where to go in order to move out of sight of the younger child. However the younger child does not forget about him when he is out of sight but is easily able (thanks to the help of a wheeled 'walker') to alter her orientation to get him back in view.

How a child moves from the earlier set of competences to the later set, is a question that can only be answered when we have a good theory of what sorts of information processing architectures are possible, and how they can modify themselves by building new layers of competence, in the process of interacting with a rich environment -- partly, though not entirely, under the control of the genome, as outlined in [Chappell & Sloman 2007](#)).

The ability to be able to model such transitions in robots is still far beyond our horizon, despite all the shallow demonstrations of 'progress' in robot training scenarios.

Kinds of dynamical system:

Moved to a separate file (10 Aug 2012)

Replaced by a more up to date version:

<http://www.cs.bham.ac.uk/research/projects/cogaff/misc/multipic-challenge.pdf>

A Multi-picture Challenge for Theories of Vision Including a section on types of dynamical system relevant to cognition.

CONTENTS

Some relevant presentations and papers

Example presentations and papers on this this topic written over the last 50 years, especially since the early 1990s.

PRESENTATIONS (PDF)

- <http://www.cs.bham.ac.uk/research/projects/cogaff/81-95.html#43>
The primacy of non-communicative language (1979)
- <http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#talk7>
7: When is seeing (possibly in your mind's eye) better than deducing, for reasoning?
- <http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#talk27>
Talk 27: Requirements for visual/spatial reasoning
- <http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#glang>
Aaron Sloman, Talk 52: What evolved first and develops first in children:
Languages for communicating? or Languages for thinking?
(Generalised Languages: GLs), 2007,
Presentation given to Birmingham Psychology department.
School of Computer Science, University of Birmingham. Work done with Jackie Chappell.
For a later more compact presentation on evolution of language and functions of vision,
see: [Talk 111](#) (below).
- <http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#talk111>
Talk 111: Two Related Themes (intertwined)
What are the functions of vision? How did human language evolve?
(Rich structured languages are needed for internal information processing, including
visual processing, representing intentions and plans, formulating questions, understanding
failures, etc. These internal languages must have evolved long before languages for
communication.)
- <http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#talk56>
Talk 56: Could a Child Robot Grow Up To be A Mathematician And Philosopher?
- <http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#talk63>
Talk 63: Kantian Philosophy of Mathematics and Young Robots
Could a baby robot grow up to be a Mathematician and Philosopher?
- <http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#talk67>
Talk 67: Why (and how) did biological evolution produce mathematicians?
OR If learning mathematics requires a teacher, where did the first teachers come from?
OR A New Approach to Philosophy of Mathematics:
Design a young explorer, able to discover "toddler theorems"
- <http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#talk79>

Talk 79: If learning maths requires a teacher, where did the first teachers come from?

- <http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#talk90>
Talk 90: Piaget (and collaborators) on Possibility and Necessity
And the relevance of/to AI/Robotics/mathematics (in biological evolution and development)
- <http://www.cs.bham.ac.uk/research/projects/cogaff/talks/#talk93>
Aaron Sloman,
What's vision for, and how does it work?
From Marr (and earlier) to Gibson and Beyond, 2011,
Online tutorial presentation, also at <http://www.slideshare.net/asloman/>

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