School of Computer Science University of Birmingham UK

Every intelligent ghost must contain a machine — an information-processing machine

Problem: Areas formed by intersecting circles

(Solved on another page.)

Aaron Sloman

http://www.cs.bham.ac.uk/~axs/

School of Computer Science, University of Birmingham

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This paper is available in two formats:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/multicirc-problem.html
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/multicirc-problem.pdf

A partial index of discussion notes in this directory is in
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/AREADME.html

Introduction

Manfred Kerber introduced me to this problem. He had learnt about it from Colin Rowat.

I'll state the problem below, and present a solution in a separate file.

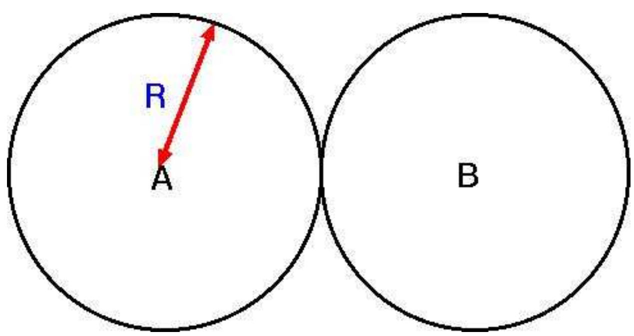

Fig 1 Two circles with centres at **A** and **B** of radius **R** touching as shown.
If two circles touch at a point (i.e. each is tangent to the other at that point)
then their centres and that point are co-linear. How do you know that's true?

Two touching circles, and a third circle passing through their point of contact, are used to present a problem, below. A separate file shows how to solve the problem, in a surprising way.

Problem statement

In what follows each circle is referred to by the letter used to indicate the centre.

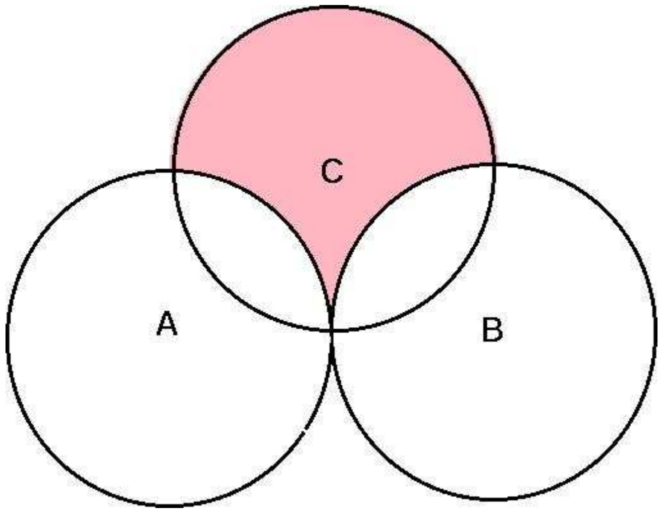

Fig 2 Add a third circle with centre **C**, also of radius **R**, above the line through **A** and **B**, with **C** placed symmetrically in relation to circles **A** and **B**, and passing through the point of contact of circles **A** and **B**, as shown. The line **AB** must then be a tangent to **C**. Why?

Note: The symmetry specified in Fig 2 implies that the centre of circle C is perpendicularly above the line joining the centres of the circles A and B. The centre of each circle must be a distance **R** from the intersection point, since they all have the same radius: **R**.

QUESTION
What is the area of the portion of circle **C** in Fig 2 that is outside the circles **A** and **B**, i.e. the area of the darker region?

That looks like a difficult question to answer because of the peculiar shape of the darker region. It is bounded by a convex curved portion at the top of circle **C**, and two concave portions below, meeting at a pointed cusp, where circles **A**, **B** and **C** intersect. No standard formula for computing areas in Euclidean geometry is directly applicable to this shape.

A solution to the problem can be found here:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/multicirc.html
There are other online discussions of this problem, considered as an exercise in geometric reasoning.

The remainder of this document raises some general issues related to research on mathematical cognition and problems in philosophy of mathematics and philosophy of mind loosely related to this problem. It does not provide the solution. For that, follow the link above.

DISEMBODIED MATHEMATICAL COGNITION

After Manfred Kerber mentioned this problem, during a conversation about mathematics, I thought about it without drawing pictures, typing, writing, or performing any other physical action related to it. After I had worked out most of the solution, in my head, as is typical of much mathematical thinking, I used a linux graphical tool (tgif) to produce various versions of the diagrams eventually used in the presentation of my solution. (There are probably several online presentations of solutions, using different forms of reasoning.)

Reflecting on this discovery process reminded me of an experience when I was a maths student at the University of Cape Town, learning about set theory (1956 or 1957). On reading the statement of the (Cantor-)Schroeder-Bernstein theorem

> A and B are finite or infinite sets, if there’s a bijection from set A **into** set B and a bijection from B **into** A, there must be a bijection between the whole of A and the whole of B -- i.e. a bijection from A **onto** B)

I thought it sounded obviously true and decided to prove it myself, instead of reading the proof.

It is easy to prove for finite sets (interpreting "into" so as not to exclude "onto") but not so easy for infinite sets, e.g. set A could be the set of rational numbers between 1 and 2, and set B the set of rational numbers between 10 and 20.

> If you don’t find it obvious that the rationals between 10 and 20 can be projected onto a subset of the rationals between 1 and 2, just work out how to project rationals between 10 and 20 onto rationals between 1.1 and 1.2, using subtraction, division and addition.

I spent most of the next few days flat on my back on my bed with my eyes shut, and eventually found the standard "diagram-based" proof, as I am sure many other mathematics students have done:
https://en.wikipedia.org/wiki/Schr%C3%B6der%E2%80%93Bernstein_theorem

If the phrase "embodied cognition" has any clear meaning, it should be obvious that that process of discovery was not an example: it was a case of disembodied cognition, as was my original process of working out the shaded area in Fig 2 reported in
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/multicirc.html.

Many years after my encounter with the Schroeder-Bernstein theorem I started hearing a subset of philosophers, cognitive scientists and AI researchers claiming that cognition must be embodied. Many of them used toy robot demos, e.g. passive walker robots, like this one
https://www.youtube.com/watch?v=N64KOQkbyiI, as evidence. I found it hard to believe how low the level of argument used by many intelligent people had sunk. None of them seemed interested in what would happen if a brick was placed in the path of the robot.

In The Computer Revolution in Philosophy (1978) I had previously pointed out that mathematical reasoning often makes use of external, e.g. hand drawn, diagrams, or calculations on written on a physical surface, for thinking about problems that are too complex to solve in short term visual memory, e.g. near the end of Section 7.3.

But that does not prove that all cognition is external or embodied, and for good reasons: much mathematical cognition cannot be because it deals with mappings between infinite sets that cannot be constructed in our environment. Even 3+2=5 implicitly refers to infinitely many triples of sets.

There are many other arguments, including the ability to read, understand, or invent stories about imaginary individuals, in imaginary locations, performing imaginary feats. Imagined individuals may have bodies quite unlike our own, including multiple arms, wings, additional eyes facing backward, etc.

Even processes of planning actions in this world can involve working mentally through merely imagined actions that are thereby discovered to be potentially fatal.

Of course, for some mathematical reasoning, such as long divisions or multiplications of multi-digit numbers, it is normal to require external aids, though there are individuals with uncanny abilities to perform complex mathematical tasks without external aids, including Ramanujan.

Aspects of diagrammatic reasoning were recently discussed in this tutorial at the Diagrams-18 conference in Edinburgh, including a brief explanation of why biological evolution necessarily produced increasingly disembodied forms of information processing, as complexity of organisms, complexity of their environments, and complexity of their needs increased:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/diagrams-tutorial.html

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META-QUESTIONS LOOSELY RELATED TO THE CIRCLES PROBLEM

This section will probably be moved to another document later.

What sorts of brain mechanism can explain:

- the ability to understand problems and proofs, e.g. the above multi-circle problem and the (diagrammatic) proof given in the accompanying document?
- the ability to discover problems like the multi-circle problem above?
- the ability to discover proofs?

What can such examples teach us about mathematical consciousness and the mechanisms it uses in humans?
For more on the analysis of consciousness see
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/family-resemblance-vs-polymorphism.html
And Chapters 6, 7 and 10 of The Computer Revolution in Philosophy, e.g. Section 10.2:
http://www.cs.bham.ac.uk/research/projects/cogaff/crp/#10.2
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/unconscious-seeing.html
(Seeing something without realising you've seen it till a later question redirects your attention to previously unnoticed details in your memory.)

Human and non-human spatial reasoning capabilities (e.g. squirrels defeating "squirrel-proof" bird feeders), including examples of spatial reasoning on this web site, challenge current psychological and neural theories about how brains work and also challenge overblown claims about the scope of current AI, including claims about the powers of "deep learning" mechanisms. More such problems are discussed in the items listed below.

Geometrical and topological problems are neither empirical, nor solvable using logic and definitions, and they involve notions of necessity and impossibility that are beyond the representational powers of statistics-based probabilistic reasoners. Necessity and impossibility are not special (extreme) cases of probability. This is one of the reasons why those who hope human intelligence can be replicated using statistics based "Deep Learning" mechanisms will be disappointed. (Could a deep learning system learn how to design deep learning systems? What would the training data look like?)

Some readers will be aware that I am merely repeating, in different words, the points made about the nature of mathematical knowledge by Immanuel Kant in his *Critique of Pure Reason* (1781), online here:
http://archive.org/details/immanuelkantscri032379mbp A tutorial summary of his claims is in #Sloman (1965).

Kant raised important questions about knowledge of geometry and arithmetic, and offered some relevant ideas, though he lacked our present (but still inadequate) understanding of information processing in brains. He suggested that

> "...our understanding in regard to phenomena and their mere form, is an art, hidden in the depths of the human soul, whose true modes of action we shall only with difficulty discover and unveil."

I suspect that if he had lived two centuries later, he would have been a leading AI researcher. My 1962 DPhil thesis was an attempt to defend Kant's philosophy of mathematics against common criticisms (e.g. criticisms referring to Einstein's theory of general relativity and Eddington's evidence that physical space is curved, i.e. non-Euclidean). But at that time I knew nothing about programming or AI, which I later came to think could be used to model a Kantian mathematician. That is not yet possible in current AI.

> In 2016, thanks to much help from Luc Beaudoin, the thesis was retyped (in India) from a pdf version produced by the Oxford University Library in 2007, and is now available here:
> http://www.cs.bham.ac.uk/research/projects/cogaff/62-80.html#1962

In 1899, the great mathematician, David Hilbert, published "The Foundations of Geometry", a new presentation of Euclidean geometry using a logic-based formal axiomatisation (and also making use of the numerical model of geometry based on coordinates).
http://www.gutenberg.org/ebooks/17384

Many mathematicians (and philosophers of mathematics) regard Hilbert's book as specifying the topic of ancient research by Euclid and others. (Unlike Frege, who raised strong objections.)

But it is arguable that Hilbert's claim to be presenting the foundations of geometry as studied by ancient mathematicians cannot be right since those mathematicians knew nothing of modern logic and there is no evidence that biological brains contain something like a logical theorem prover.

Moreover, there were constructions known to ancient mathematicians, even at the time of Euclid, that were excluded from Euclid's *Elements* for non-mathematical reasons (e.g. lack of elegance?), and Hilbert's axiomatisation did not include them. One of them was the *"neusis"* construction, that can be used to trisect an arbitrary angle, as shown here (a challenge provably impossible in Euclidean geometry as represented by Hilbert):

http://www.cs.bham.ac.uk/research/projects/cogaff/misc/trisect.html

So there were aspects of ancient geometrical reasoning that Hilbert had not captured in his system.

As far as I know, the neusis construction is not mentioned, and is not consistent with Hilbert's axioms (from which the impossibility of angle trisection, for arbitrary angles could be derived).

That raises questions about how ancient brain mechanisms were able to produce such amazing mathematical results, many still in constant use world-wide, e.g. Pythagoras' Theorem.
https://en.wikipedia.org/wiki/Pythagorean_theorem

A possible defense of Hilbert?

I doubt that Hilbert thought his work on foundations of geometry had any connection with scientific explanations of how brains make mathematical discoveries.

Perhaps his work could be regarded as presenting a "rational reconstruction" of Euclid's *Elements*, namely a presentation of what Euclid (and other ancient mathematicians) should have done when studying and teaching mathematics. I suspect there are many logicians, mathematicians and philosophers who explicitly or implicitly hold such a view, which implies that the ancient mathematicians were not *really* doing mathematics: they were thinking intelligently about spatial structures, objects, relationships and processes, and they managed, by luck, with the aid of good guessing mechanisms provided by evolution, to stumble across some mathematical truths, but they never really understood how those truths could be *proved* to be correct, as opposed to merely being effective at convincing and teaching other individuals?

An alternative position is that there just happen to be different types of mathematical domain, requiring different kinds of mathematical discovery mechanism. For example: mechanisms based on logic and algebra make explicit use only of discrete forms of representation, allowing only discrete changes (e.g. concatenation, re-ordering, substitution, adding or removing parts, etc.), whereas the mechanisms involved in geometrical reasoning of the sort I learnt at school involve continuous deformations, including, translation, rotation, stretching, folding, changing 2D or 3D curvature (e.g. along a spiral curve, or on the surface of an egg, or torus).

But that still leaves questions about the nature and status of those ancient (biological) discovery mechanisms:

- What were the discovery mechanisms?
- How did they work so well?
- How did biological evolution come to discover/produce them?
  E.g. what were their evolutionary precursors, and to what extent are those precursor mechanisms still used in other species?
- How do they develop in individuals, since they don't seem to be present at birth?
- What brain mechanisms are required to support them?
- Could they be implemented in virtual machines running on computers?
- Would using them in our attempts to design intelligent robots produce better results than currently used mechanisms (e.g. based on logic, rule-based systems, artificial neural nets)?
- What sorts of research activities are needed to identify the mechanisms (and all their variants) and to identify their powers and limitations?

- Can those mechanisms be replicated in robots using digital computers instead of biological brain mechanisms?

For further discussion of the possibility of a different kind of "Super Turing" virtual machine using deformable surfaces instead of discrete labels in discrete locations, see
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/super-turing-geom.html
Some philosophical background is here:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/super-turing-phil.html

A NOTE ABOUT THE CONTEXT OF THIS DOCUMENT

This is part of the Meta-Morphogenesis project:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/meta-morphogenesis.html

A partial, incomplete, discussion of possible extensions to our current ideas of computation is in another paper:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/super-turing-geom.html
I think Alan Turing was exploring related ideas around the time of his death in 1954.

Many more examples can be found in the items listed below.

Reminder: A solution to the intersecting circles area problem can be found here:
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/multicirc.html

RELATED DOCUMENTS

http://www.cs.bham.ac.uk/research/projects/cogaff/misc/mathstuff.html
Mathematical phenomena, their evolution and development
(Examples and discussions on this web site.)

http://www.cs.bham.ac.uk/research/projects/cogaff/misc/impossible.html
Some (Possibly) New Considerations Regarding Impossible Objects

http://www.cs.bham.ac.uk/research/projects/cogaff/misc/toddler-theorems.html
Meta-Morphogenesis and Toddler Theorems: Case Studies

http://www.cs.bham.ac.uk/research/projects/cogaff/misc/triangle-sum.html
The Triangle Sum Theorem
Old and new proofs concerning the sum of interior angles of a triangle.
(More on the hidden depths of triangle qualia.)

http://www.cs.bham.ac.uk/research/projects/cogaff/misc/torus.html
Reasoning About Continuous Deformation of Curves on a torus and other things.

http://www.cs.bham.ac.uk/research/projects/cogaff/movies/ijcai-17/small-pencil-vid.webm
A 17 month toddler discovers and solves a 3D topology problem

http://www.cs.bham.ac.uk/research/projects/cogaff/misc/super-turing-geom.html
A Super-Turing (Multi) Membrane Machine for Geometers
(Also for toddlers, and other intelligent animals)

What is information?
http://www.cs.bham.ac.uk/research/projects/cogaff/misc/austen-info.html
The novelist Jane Austen’s (implicit) answer is compared with the mathematician Claude Shannon’s answer, which has confused scientists and philosophers for several decades. (Shannon himself was not confused: he merely used unfortunate terminology.)

Aaron Sloman, 1965, "Necessary", "A Priori" and "Analytic", *Analysis*, Vol 26, No 1, pp. 12--16, http://www.cs.bham.ac.uk/research/projects/cogaff/62-80.html#1965-02

Maintained by Aaron Sloman
School of Computer Science
The University of Birmingham